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UNIVERSITY OF NORTH BENGAL

MASTERS OF SCIENCE-MATHEMATICS

SEMESTER -I

DIFFERENTIAL GEOMETRY

DEMATH1ELEC4

BLOCK-1

UNIVERSITY OF NORTH BENGAL

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FOREWORD

The Self-Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours.

DIFFERENTIAL GEOMETRY

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BLOCK 1 DIFFERENTIAL GEOMETRY

Introduction to the Block

In this block we will go through differential Geometry…Cartography & Differential Geometry , Differential Geometry And Differential Topology , Vector Fields and Flows… . . Vector Fields , Vector Bundles And Submersions , Geodesics , Convexity , Curvature………………

Unit I Deals with Differential Geometry Cartography & Differential Geometry

Unit II Deals with Differential Geometry And Differential Topology

Unit III Deals with Vector Fields and Flows… . . Vector Fields

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Unit V Deals with Geodesics

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UNIT-I: DIFFERENTIAL GEOMETRY

STRUCTURE

1.0 Objectives

1.1 Introduction

1.2 Differential Geometry … Cartography And Differential Geometry

1.3 Curves

1.4 Basics Of Euclidean Geometry

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1.0 OBJECTIVES

After studying this unit , you should be able to:

- Understand about Differential Geometry
- Cartography And Differential Geometry   Curves
- Basics Of Euclidean Geometry

1.1 INTRODUCTION

Differential geometry arose and developed as a result of and in connection to the mathematical analysis of curves and surfaces Mathematical analysis of curves and surfaces had been developed to answer some of unanswered questions that appeared in calculus like the reasons for relationships between complex shapes and curves , series and analytic functions Differential Geometry … Cartography And Differential Geometry , Curves , Basics Of Euclidean Geometry

1.2 DIFFERENTIAL GEOMETRY

WHAT IS DIFFERENTIAL GEOMETRY?

CARTOGRAPHY AND DIFFERENTIAL GEOMETRY

Carl Friedrich GauB (1777 - 1855) is the father of differential geometry . He was (among many other things) a cartographer and many terms in modern differential geometry (chart , atlas , map , coordinate system , geodesic , etc .) reflect these origins . He was led to his Theorema Egregium by the question of whether it is possible to draw an accurate map of a portion of our planet . Let us begin by discussing a mathematical formulation of this problem .

Consider the two dimensional sphere S2 sitting in the three dimensional Euclidean space R3 . It is cut out by the equation correspondence with a small region in the plane z = 0 . In this book we will represent this with the notation 0 : U ^ 0 (U) C R2 and call such an object a chart or a system of local coordinates .

What does it mean that 0 is an "accurate" map? Ideally the user would want to use the map to compute the length of a curve in S2 . The length of a curve 7 connecting two points p , q □ S2 is given by the formula

so the user will want the chart 0 to satisfy L (7) = L (0 o 7) for all curves 7 . It is a consequence of the Theorema Egregium that there is no such chart .

Perhaps the user of such a map will be content to use the map to plot the shortest path between two points p and q in U . This path is called a geodesic . Denote this shortest path by ypq . It satisfies L (ypq) = du (p, q) where

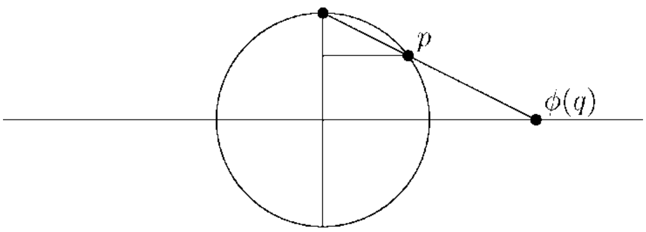
du (p , q) = inf {L (7) | 7 (t) e U , 7 (0) = p , 7 (1) = q} so our less demanding user will be content if the chart 0 satisfies du (p , q) = dE (0 (p) , 0 (q))

where dE (p , q) is the length of the shortest path . It is also a consequence of the Theorema Egregium that there is no such chart .

Now suppose our user is content to have a map which makes it easy to navigate along the shortest path connecting the two paths . Ideally the user would use a straight edge , magnetic compass , and protractor to do this . S / he would draw a straight line on the map connecting p and q and steer a course which maintains a constant angle (on the map) between the course and meridians . This can be done by the method of stereographic projection . This chart is conformal (which means that it preserves angles) . According to Wikipedia stereographic projection was known to the ancient Greeks and a map using stereographic projection was constructed in the early 16th century . stereographic projection; the latter exercise deals with the Poincare model of the hyperbolic plane . The hyperbolic plane provides a counter example the Euclid's Parallel Postulate .

Exercise . It is more or less obvious that for any surface M C R3 there is a unique shortest path in M connecting them if they are sufficiently close . This shortest path is called the minimal geodesic connecting p and q . Use this fact to prove that the minimal geodesic joining two points p and q in S2 is an arc of the great circle through p and q . (This is the intersection of the sphere with the plane through p , q , and the center of the sphere .) Also prove that the minimal geodesic connecting

n



Stereographic Projection

Notes

two points in a plane is the straight line segment connecting them . Hint: Both a great circle in a sphere and a line in a plane are preserved by a reflection .

Exercise . Stereographic projection is defined by the condition that for p e S2 \ n the point 0 (p) lies in the xy - plane z = 0 and the three points n = (0 , 0 , 1) , p , and 0 (p) are collinear . Using the formula that the cosine of the angle between two unit vectors is their inner product prove that 0 is conformal . Hint: The plane of p , q , and n intersects the xy - plane in a straight line and the sphere in a circle through n . The plane of n , p , and 0 (p) intersects the sphere in a meridian . A proof that stereographic projection is conformal can be found . The proof is elementary in the sense that it doesn't use calculus .

Exercise . It may seem fairly obvious that you can't draw an accurate map of a portion of the earth because the sphere is curved . However the cylinder C = { (x , y , z) e R3 | x2 + y2 = 1} is also curved , but the map 0 : R2 ^ C defined by 0 (s , t) = (cos t , sin t , s) preserves lengths of curves , i . e . L (0 o y) = L (y) for any curve y : [a , b] ^ R2 . Prove this .

Standard Notations . The standard notations N , No , Z , Q , R , C denote respectively the natural numbers (= positive integers) , the non - negative integers , the integers , the rational numbers , the real numbers , and the complex numbers . We denote the identity map of a set X by idX and the n x n identity matrix by Hn or simply 1 . The notation V* is used for the dual of a vector space V , but when K is a field like R or C the notation K* is sometimes used for the multiplicative group K \ {0} . The terms smooth , infinitely differentiable , and C^ are all synonymous .

Coordinates

The rest of this chapter defines category of smooth manifolds and smooth maps between them . Before giving the precise definitions we will introduce some terminology and give some examples .

Definition . A chart on a set M is a pair (f , U) where U is a subset of M and f : U ^ f (U) is a bisection1 from U to an open set f (U) in Rm . An atlas on M is a collection A = { (fa , Ua) }aGA of charts such that the domains Ua cover M , i . e .

M = J Ua .

ag A

The idea is that if f (p) = (x1 (p) , . . . , xm (p)) for p e U then the functions Xi form a system of local coordinates defined on the subset U of M . The dimension of M should be m since it takes m numbers to uniquely specify a point of U . We will soon impose conditions on charts (f , U) , however for the moment we are assuming nothing about the maps f (other than that they are bijective) .

Example . Every open subset U C Rm has an atlas consisting of a single chart , namely (f , U) = (idu , U) where id^ denotes the identity map of U .

Example . Assume that W C Rm and V C Rn are open sets , that M is a subset of the product Rm x Rn = Rm + n , and f : W ^ V is a map whose graph is a subset of M , i . e .

graph (f) := { (x , y) e W x V | x e W , y = f (x) } C M .

Let U = (W n V) n graph (f) and let f (x , y) = x be the projection of U onto W . Then the pair (f , U) is a chart on M . The inverse map is given by f - 1 (x) = (x , f (x))) .

Example . The m - sphere

Sm = {p = (xo , . . . , xm) e Rm + 1 | x2 +   + xm = 1}

has an atlas consisting of the 2m + 2 charts 0j± : Ui± ^ Dm where Dm is the open unit disk in Rm , Uj± = {p £ Sm | ± Xj > 0} , and 0j± is the projection which discards the ith coordinate .

Notes

Example . Let A = AT £ R (m + 1) x (m + L be a symmetric matrix and define a quadratic form F : Rm + 1 ^ R by

F (p) := xtAx , p = (x0 , . . . , xm) .

After a linear change of coordinates the function F has the form

f (p) = x0 + ■ ■ ■ + xk— xk + i — ■ ■ ■— x2 .

(Here r is the rank of the matrix A .) The set M = F - 1 (1) has an atlas of 2m + 1 charts by the same construction as in Example , in fact Sm + 1 is the special case where A =]ln , the n x n identity matrix .

enumerates the familiar quadric surfaces in R3 . When W = R2 and V = R the paraboloids are examples of graphs and the ellipsoid and the two hyperboloids are instances of the quadric surfaces defined . The sphere is an instance of the ellipsoid (a = b = c =1) and the cylinder is a limit (as c ^ rc>) of the hyperbolic paraboloid . The pictures were generated by computer using the parameterizations

x = acos (f) sin (s) , y = bsin (t) sin (s) , z = ccos (s)

for the ellipsoid ,

x = a cos (f) sinh (s) , y = b sin (t) sinh (s) , z = c cosh (s)

for the hyperbolic paraboloid , and

x = a cosh (f) sinh (s) , y = b sinh (t) sinh (s) , z = c cosh (s)

for the elliptic paraboloid . These quadric surfaces will be often used in the sequel to illustrate important concepts .

In the following two examples K denotes either the field R of real numbers or the field C of complex numbers , K* := {A £ K | A = 0} denote the corresponding multiplicative group , and V denotes a vector space over K .

2 , 2 , 2 i

x + y + z =1

x2 y2 z2 + - —I= 1

a2 b2 c2 1

x2 I y2 = 1

Example . The projective space of V is the set of lines (through the origin) in V . In other words ,

P (V) = {t C V 1t is a 1 - dimensional K - linear subspace}

When K = R and V = Rm + 1 this is denoted by RPm and when K = C and C = Rm + 1 this is denoted by CPm . For our purposes we can identify the spaces Cm + 1 and R2m±2 but the projective spaces CPm and RP2m are very different . The various lines t £ P (V) intersect in the origin , however , after the harmless identification

P (V) = {[v] | v £ V \ {0} , [v] := K*v = Kv \ {0}

the elements of P (V) become disjoint , i . e . P (V) is the set of equivalence classes of an equivalence relation on the open set V \ {0} . Assume that V = Km + 1 and define an atlas on P (V) as follows . For each i = 0 , 1 , . . . , m let Ui = {[v] | , v = (x0 , . . . , xm) Xi = 0} and define a bijection 0 : U ^ Km by the formula

0i ([v]) = (X0 , . . . , Xi - 1 , Xi±i . . . , xm^ .

\ / y> . ry . ry . ry ■ }

\ xi xi xi xi J

This atlas consists of m + 1 charts .

Example . For each positive integer k the set

Gk (V) := {t C V 11 is a k - dimensional K - linear subspace}

is called the Grassmann manifold of k - planes in V . Thus G1 (V) = P (V) . Assume that V = Kn and define an atlas on Gk (V) as follows . Let e1 , . . . , en be the standard basis for Kn , i . e . ei is the ith column of the n x n identity matrix 1n . Each partition {1 , 2 , . . . , n} = I U J , I = {i1 < ••• < ik} , J = j1 < ■ ■ ■ < jn - k of the first the first n natural numbers determines a direct sum decomposition

Kn = V = V / □ Vj

Notes

via the formulas V / = Kei1 + ■ ■ ■ + Keifc and VJ = Kej1 + ■ ■ ■ + Kejn_fc . Let U / denote the set of t £ Gk (V) which are transverse to VJ , i . e . such that t n VJ = {0} . The elements of U / are precisely those k - planes of form t = graph (A) where A : V / ^ Vj is a linear map . Define 0 / : Ui ^ Kfcx (n_fc) by the formula

n - k

0 / (t) — (ars) , Aeir — ^ ^ asrejs .

s=1

Exercise . Prove that the set of all pairs (0 / , U /) as I ranges over the subsets of {1 , . . . , n} of cardinality k form an atlas .

Topological Manifolds*

Definition . A topological manifold is a topological space M such that each point p e M has an open neighborhood U which is homeomorphic to an open subset of a Euclidean space .

Brouwer's Invariance of Domain Theorem asserts that , when U C Rm and V C Rn are nonempty open sets and f : U ^ V is a homeomorphism , then m = n . This means that if M is a connected topological manifold and some point of M has a neighborhood homeomorphic to an open subset of Rm , then every point of M has a neighborhood homeomorphic to an open subset of that same Rm . In this case we say that M has dimension m or is m - dimensional or is an m - manifold . Brouwer's theorem is fairly difficult but if f is a diffeomorphism the result is an easy consequence of the invariance of the rank in linear algebra and the chain rule .

By definition , a topological m - manifold M admits an atlas where every chart (f , U) of the atlas is a homeomorphism f : U ^ f (U) from an open set of U C M to an open set f (U) C Rm . The following definition and lemma explains when a given atlas determines a topology on M .

Definition . Let M be a set . Two charts (fi , Ui) and (f2 , U2) are said to be topologically compatible iff f1 (U1 n U2) is open in f1 (U1) , f2 (U1 n U2) is open in f2 (U2) , and the transition map

f21 := f2 ° f - 1 : f 1 (U1 n U2) ^ f2 (U1 n U2)

is a homeomorphism . An atlas is said to be a topological atlas iff any two charts in this atlas are topologically compatible .

Lemma . Let A = { (fa , Ua) }aeA be an atlas on a set M . Then

The collection of all subsets U C M such that fa (U n Ua) is an open subset of Rm is a topology on M and M is a topological manifold in this topology .

If M is a topological manifold and A is an atlas for M such that each fa is a homeomorphism , then the topology in part (a) coincides with the topology of M .

If M is already a topological manifold , then the collection of all charts (U , f) on M such that f is a homeomorphism is a topological atlas . It is the unique maximal atlas in the sense that it contains every other topological atlas . However , we will often need to consider smaller atlases , even finite atlases that any atlas determines the topology of M .

Exercise . Equip each with a topology by showing that the atlas in the example is a topological atlas . Conclude that each of these examples is a topological manifold . The Grassmann Manifold is tricky because you need an explicit formula for the transition map .

Any subset S C X of a topological space X inherits a topology from X , called the relative topology of S . A subset U0 C S is called relatively open in S (or S - open for short) if there is an open set U C X such that U0 = U fl S . A subset A0 C S is called relatively closed (or S - closed for short) if there is a closed set A C X such that Ao = A f S . The relative topology on S is the finest topology such that the inclusion map S ^ X is continuous .

Exercise . Show that the relative topology satisfies the axioms of a topology (i . e . arbitrary unions and finite intersections of S - open sets are S - open , and the empty set and S itself are S - open) . Show that the complement of an S - open set in S is S - closed and vice versa .

Exercise Each of the sets defined is a subset of some Euclidean space Rk . Show that the topology in Exercise is the relative topology inherited from the topology of Rk . 2

If ~ is an equivalence relation on a topological space X , the quotient space

Y := X / ~ := {[x] | x e X} is the set of all equivalence classes [x] := {x' e X | x' ~ x} . The map

n : X ^ Y

defined by n (x) = [x] will be called the obvious projection . The quotient space inherits the quotient topology from Y . Namely , a set V C Y is open in this topology iff the preimage n - 1 (V) is open in X . This topology is the coarsest topology on Y such that projection n : X ^ Y is continuous . Since the operation V ^ n - 1 (V) commutes with arbitrary unions and inter - sections the quotient topology obviously satisfies the axioms of a topology .

Exercise . Show that the atlases for RPm and CPm defined in equip P (V) with the quotient topology inherited from the open set V \ {0} . (Recall that in that exercise V = Km and K = R or C .)

Smooth Manifolds Defined*

Let U C Rn and V C Rm be open sets . A map f : U ^ V is called smooth iff it is infinitely differentiable , i . e . iff all its partial derivatives

da1 + + an f

a = (ai , . . . , ak) e N£ ,

dxa1 • • • dx°n

exist and are continuous . In later chapters we will sometimes write C™ (U , V) for the set of smooth maps from U to V .

Definition . Let U C Rn and V C Rm be open sets . For a smooth map f = (f1 , . . . , fm) : U ^ V and a point x e U the derivative of f at x is the linear map df (x) : Rn ^ Rm

This linear map is represented by the Jacobian matrix of f at x which will also be denoted by

(f (x)

dxi ( )

df (x) :=

d fm

\ dx1

Note that we use the same notation for the Jacobian matrix and the corresponding linear map from Rn to Rm .

The derivative satisfies the chain rule . Namely , if U C Rn , V C Rm , W C R^ are open sets and f : U ^ V and g : V ^ W are smooth maps then g o f : U ^ W is smooth and

d (g o f) (x) = dg (f (x)) o df (x) : Rn ^ R^

for every x e U . Moreover the identity map id^ : U ^ U is always smooth and its derivative at every point is the identity map of Rn . This implies that , if f : U ^ V is a diffeomorphism (i . e . f is bijective and f and f - 1 are both smooth) , then its derivative at every point is an invertible linear map . This is why the Invariance of Domain Theorem is easy for diffeomorphisms: if f : U ^ V is a diffeomorphism , then the Jacobian matrix df (x) e Rmxn is invertible for every x e U and so m = n . The Inverse Function Theorem is a kind of converse .

Definition (Smooth Manifold) . Let M be a set . A chart on M

is a pair (4> , U) where U C M and $ is a bijection from U to an open subset $ (U) C Rm of some Euclidean space . Two charts ($1 , U1) and ($2 , U2) are said to be smoothly compatible iff $1 (U1 n U2) and $2 (U n U2) are both open in Rm and the transition map

$21 = $2 ° $ - 1 : $i (Ui n U2) ^ $2 (Ui n U2)   (1 - 4 - 2)

is a diffeomorphism . A smooth atlas on M is a collection A of charts on M any two of which are smoothly compatible and such that the sets U , as ($, U) ranges over A , cover M (i . e . for every p e M there is a chart ($, U) e A with p e U) . A maximal smooth atlas is an atlas which contains every chart which is smoothly compatible with each of its members . A smooth manifold is a pair consisting of a set M and a maximal atlas A on M .

Lemma . If A is an atlas , then so is the collection A of all charts compatible with each member of A . The atlas A is obviously maximal . In other words , every atlas extends uniquely to a maximal atlas .

Proof . Let ($1 , U1) and ($2 , U2) be charts in A and let x e $1 (U1 n U2) . Choose a chart ($, U) e A such that $ - 1 (x) e U - Then $1 (U n U1 n U2) is an open neighborhood of x in Rm and the transition maps

$ o $ - 1: $1 (u n U1 n U2) ^ $ (U n U1 n U2) ,

$2 o $ - 1: $ (U n U1 n U2) ^ $2 (U n U n U2)

are smooth by definition of A - Hence so is their composition . This shows that the map $2 o $ - 1 : $1 (U1 n U2) ^ $2 (U1 n U2) is smooth near x . Since x was chosen arbitrary , this map is smooth - Apply the same argument to its inverse to deduce that it is a diffeomorphism - Thus A is an atlas - □

Definitions are mutatis mutandis the same , so every smooth atlas on a set M is a fortiori a topological atlas , i - e - every smooth manifold is a topological manifold Moreover the definitions are worded in such a way that it is obvious that every smooth map is continuous -

Exercise . Show that each of the atlases from the examples in is a smooth atlas

When A is a smooth atlas on a topological manifold M one says that A is a smooth STRUCTURE on the (topological) manifold M iff A C B , where B is the maximal topological atlas on M . When no confusion can result we generally drop the notation for the maximal smooth atlas as in the following exercise .

Exercise . Let M , N , and P be smooth manifolds and f : M ^ N and g : N ^ P be smooth maps . Prove that the identity map idM is smooth and that the composition g o f : M ^ P is a smooth map . (This is of course an easy consequence of the chain rule

Remark . It is easy to see that a topological manifold can have many distinct smooth STRUCTUREs . For example , { (idR , R) } and { (<f , R) } where 0 (x) = x3 are atlases on the real numbers which extend to distinct smooth STRUCTUREs but determine the same topology . However these two manifolds are diffeomorphic via the map x ^ x1 / 3 . In the 1950's it was proved that there are smooth manifolds which are homeomorphic but not diffeomorphic and that there are topological manifolds which admit no smooth STRUCTURE . In the 1980's it was proved in dimension m = 4 that there are uncountably many smooth manifolds that are all homeomorphic to R4 but no two of them are diffeomorphic to each other . These theorems are very surprising and very deep .

A collection of sets and maps between them is called a category if the collection of maps contains the identity map of every set in the collection and the composition of any two maps in the collection is also in the collection . The sets are called the objects of the category and the maps are called the morphisms of the category . An invertible morphism whose inverse is also in the category is called an isomorphism . Some examples are the category of all sets and maps , the category of topological spaces and continuous maps (the isomorphisms are the homeomorphisms) , the category of topological manifolds and continuous maps between them , and the category of smooth manifolds and smooth maps (the isomorphisms are the diffeomorphisms) . Each of the last three categories is a subcategory of the preceding one .

Often categories are enlarged by a kind of "gluing process" . For example , the "global" category of smooth manifolds and smooth maps was constructed from the "local" category of open sets in Euclidean space and smooth maps between them via the device of charts and atlases . (The chain rule shows that this local category is in fact a category .) The point of Definition is to show that topological manifolds can be defined in an manner analogous to the definition we gave for smooth manifolds in Definition Other kinds of manifolds (and hence other kinds of geometry) are de - fined by choosing other local categories , i . e . by imposing conditions on the transition maps in Equation For example , a real analytic manifold is one where the transition maps are real analytic , a complex manifold is one whose coordinate charts take values in Cn and whose transition maps are holomorphic diffeomorphisms , and a symplectic manifold is one whose coor - dinate charts take values in R2n and whose transition maps are canonical transformations in the sense of classical mechanics . Thus CPn is a complex manifold and RPn is a real analytic manifold .

The Master Plan

In studying differential geometry it is best to begin with extrinsic differential geometry which is the study of the geometry of submanifolds of Euclidean space . This is because we can visualize curves and surfaces in R3 . However , there are a few topics require the more abstract definition even to say interesting things about extrinsic geometry . There is a generalization to these manifolds involving a STRUCTURE called a Riemannian metric . We will call this generalization intrinsic differential geometry . Examples fit into this more general definition so intrinsic differential geometry .

Since an open set in Euclidean space is a smooth manifold the definition of a submanifold of Euclidean space is mutatis mutandis the same as the definition of a submanifold of a manifold . The definitions in Chapter are worded in such a way that it is easy to read them either extrinsically or intrinsically and the subsequent chapters are mostly (but not entirely) extrinsic . Those sections which require intrinsic differential geom - etry (or which translate extrinsic concepts into intrinsic ones) are marked with a * .

1.3 CURVES

Definition and Examples

a cartographer and many terms in modern differential geometry (chart , atlas , map , coordinate system , geodesic A (parametrized) curve (in Euclidean space) is a mapping a: I *u* R^n , where I is an interval in the real line . We also use the notation

I 9 t —U a (t) e R^n ,

which emphasizes that a sends each element of the interval I to a certain point in R^n . We say that a is (of the class of) C^k provided that it is k times continuously differentiable . We shall always assume that a is continuous (C^0) , and whenever we need to differentiate it we will assume that a is differentiable up to however many orders that we may need .

Some standard examples of curves are a line which passes through a point p *e* R^n , is parallel to the vector v e R^n , and has constant speed ||v||

[0 , 2n] 9 t —U p + tv e R^n; a circle

of radius R in the plane , which is oriented counterclockwise ,

[0 , 2n] 9 t —U (r cos (t) , r

sin (t)) e R^2; and the right handed helix (or

corkscrew) given by

R 9 t —U (r cos (t) , r sin (t) , t) e R^3 .

Other famous examples include the figure - eight curve

[0 , 2n] 9 t —U (sin (t) , sin (2t)) e R^2 ,

the parabola

R 9 t —U (t , t^2) E R^2 ,

and the cubic curve

R *3* t —U (t , t^2 , t^3) *e* R^3 .

Exercise . Sketch the cubic curve (Hint: First draw each of the projections into the xy , yz , and zx planes) .

Notes

Exercise . Find a formula for the curve which is traced by the motion of a fixed point on a wheel of radius r rolling with constant speed on a flat surface (Hint: Add the formula for a circle to the formula for a line generated by the motion of the center of the wheel . You only need to make sure that the speed of the line correctly matches the speed of the circle) .

Exercise . Let a: I *u* R^n , and ft: J *u* R^n be a pair of differentiable curves . Show that

(a (t) , ft (t)) J = (a' (t) , ft (t)) + (a (t) , ft' (t) and

(a (t) , a' (t)

(Hint: The first identity follows immediately from the definition of the inner - product , together with the ordinary product rule for derivatives . The second identity follows from the first once we recall that || ■ || := (■ , -) $^{1/2}$) .

Exercise . Show that if a has unit speed , i . e . , ||a' (t) || = 1 , then its velocity and acceleration are orthogonal , i . e . , (a' (t) , a" (t)) = 0 .

Exercise . Show that if the position vector and velocity of a planar curve a: I *u* $_R$2

are always perpendicular , i . e . , (a (t) , a' (t)) = 0 , for all t E I , then a (I) lies on a circle centered at the origin of R^2 .

Exercise . Use the fundamental theorem of Calculus for real valued func - tions to show:

I^b

a (b) — a (a) = / a' (t) dt .

Exercise . Prove that

||a (b) — a (a) || < *f* ||a' (t) || dt .

J a

(Hint: Use the fundamental theorem of calculus and the Cauchy - Schwarts inequality to show that for any unit vector u E R^n ,

p b r b

(a (b) — a (a) , u) = (a' (t) , u) dt < I ||a' (t) ||dt .

a a

Then set u := (a (b) — a (a)) / ||a (b) — a (a) || .

The previous exercise immediately yields:

Theorem (Mean Value Theorem for curves) . If a: I ^ R^n be a C^1

curve , then for every t , s E I ,

||a (t) — a (s) || < sup ||a'|||t — s| .

[t , s]

Reparametrization

We say that fl: J ^ R^n is a reparametrization of a: I ^ R^n provided that there exists a smooth bijection 6: I ^ J such that a (t) = fl (6 (t)) . In other words , the following diagram commutes:

For instance fl (t) = (cot (2t) , sin (2t)) , 0 < t < n , is a reparametrization a (t) = (sin (t) , cos (t)) , 0 < t < 2n , with 6: [0 , 2n] ^ [0 , n] given by 6 (t) = t / 2 .

The geometric quantities associated to a curve do not change under reparametrization . These include length and curvature as we define below

Length and Arclength

By a partition P of an interval [a , b] we mean a collection of points {t_0 , . . . , t_n} of [a , b] such that

a = t*o* < t*i* < ■ ■ ■ < t_n = b .

The approximation of the length of a with respect to P is defined as

Notes

length[a , P] := ^ ||a (ti) - a (ti_*i*) || , i=1

and if Partition[a , b] denotes the set of all partitions of [a , b] , then the length of a is given by

length[a] := sup { length[a , P] | P € Partition[a , b] } ,

where 'sup' denotes the supremum or the least upper bound .

Exercise . Show that the shortest curve between any pairs of points in R^n is the straight line segment joining them . (Hint: Use the triangle inequality) .

We say that a curve is rectifiable if it has finite length .

Exercise* (Nonrectifiable curves) . Show that there exists a curve a: [0 , 1] ^ R^2 which is not rectifiable (Hint: One such curve , known as the Koch curve , may be obtained as the limit of a sequence of curves a_i: [0 , 1] ^ R defined as follows . Let a_0 trace the line segment [0 , 1] . Consider an equilateral triangle of sides 1 / 3 whose base rests on the middle third of [0 , 1] . Deleting this middle third from the interval and the triangle yields the curve traced by a*i* .

Repeating this procedure on each of the 4 subsegments of a_1 yields a_2 . Similarly a_{i+1} is obtained from a_i . You need to show that a_i converge to a (continuos) curve , which may be done using the Arzela - Ascoli theorem . It is easy to see that this limit has infinite length , because the length of a*i* is $(4 / 3)^j$. Another example of a nonrectifiable curve a: [0 , 1] ^ R^2 is given by a (t) := (t , t sin (n / t)) , when t = 0 , and a (t) := (0 , 0) otherwise . The difficulty here is to show that the length is infinite .)

If a curve is C^1 , then its length may be computed as the following theorem shows . Note also that the following theorem shows that a C^1 curve over a compact domain is rectifiable . First we need the following fact:

Theorem (Length of C^1 curves) . Show that if a: I ^ R^n is a C^1 curve , then

length[a] = *J* ||a' (t) || dt .

Proof . It suffices to show that (i) length[a , P] is not greater than the above integral , for any P G Partition[a , b] , and (ii) there exists a sequence P_N of partitions such that $\lim_{N}$^^ length[a , P_N] is equal to the integral . The first part follows quickly from Exercise 7 . To prove the second part , let P_N be a partition given by t_i := a + i (b — a) / N . Recall that , by the definition of integral , for any e > 0 , we may choose N large enough so that

Next note that the mean value theorem for curves yields that

sup ||a' (sj) | — |a' (tj) | < sup |a' (s_i) — a' (t_i ^{s}i*g*$^{[t}$i - 1 $^{, t}$i$^{]}$

Finally since a' is continuos on the closed interval [a , b] , we may suppose that N is so large that

^{su}p Il$^{a\prime (}$$s_i$$^{) —}$ <^*77* ---- w

si€[ti - i , ti] 2 (b — a)

which completes the proof .

Exercise . Compute the length of a circle of radius r , and the length of one cycle of the curve traced by a point on a circle of radius r rolling on a straight line .

Exercise (Invariance of length under reparametrization) . Show that if fl is a reparametrization of a C^1 curve a , then length[fl] = length[a] , i . e . , length is invariant under reparametrization (Hint: you only need to recall the chain rule together with the integration by substitution .)

Let L := length[a] . The arclength function of a is a mapping s: [n , h] ^ [0 , L] given by

s (t) := / ||a' (u) || du .

Thus s (t) is the length of the subsegment of a which stretches from the initial time n to time t .

Exercise (Regular curves) . Show that if a is a regular curve , i . e . , I|a' (t) || = 0 for all t *E* I , then s (t) is an invertible function , i . e . , it is one - to - one (Hint: compute s' (t)) .

Notes

Exercise (Reparametrization by arclength) . Show that every regu - lar curve a: [n , h] ^ R^n , may be reparametrized by arclength (Hint: Define : [0 , L] ^ R^n by (t) := a (s^{-1} (t)) , and use the chain rule to show that H f l ' l l = 1; you also need to recall that since f (f^{-1} (t)) = t , then , again by chain rule , we have (f^{-1}) ' (t) = 1 / f' (f^{-1} (t)) for any smooth function f with nonvanishing derivative .)

Cauchy's integral formula and curves of constant width

Let a: ^ R^2 be a curve and u (0) := (cos (0) , sin (0)) be a unit vector . The projection of a into the line passing through the origin and parallel to u is given by a_u (t) := (a (t) , u) u .

Exercise (Cauchy's integral formula) . Show that if a: I ^ R^2 has

length L , then the average length of the projections a_u , over all directions , is 2L / n , i . e . ,

1 f 2^ 2L

— length[au (0)]

d6 = — . 2n J*o* n

(Hint: First prove this fact for the case when a traces a line segment . Then a limiting argument settles the general case , once you recall the definition of length .)

As an application of the above formula we may obtain a sharp inequality involving width of closed curves . The width of a set X C R^2 is the distance between the closest pairs of parallel lines which contain X in between them . For instance the width of a circle of radius r is 2r . A curve a: [a , b] ^ R^2 is said to be closed provided that a (a) = a (b) . We should also mention that a is a C^k closed curve provided that the (one - sided) derivatives of a match up at a and b .

Exercise (Width and length) . Show that if a: [a , b] ^ R^2 is a closed curve with width w and length L , then

L

w < — . n

Note that the above inequality is sharp , since for circles w = L / n . Are there other curves satisfying this property? The answer may surprise you . For any unit vector u (0) , the width of a set X C R^2 in the direction u , w_u , is defined as the distance between the closest pairs of lines which contain X in between them . We say that a closed curve in the plane has constant width provided that w_u is constant in all directions .

Exercise . Show that if the equality in holds then a is a curve of constant width .

The last exercise would have been insignificant if circles were the only curves of constant width , but that is not the case:

Exercise (Reuleaux triangle) . Consider three disks of radius r whose centers are on an equilateral triangle of sides r , see Figure 2 . Show that the curve which bounds the intersection of these disks has constant width . Also show that similar constructions for any regular polygon yield curves of constant width .

It can be shown that of all curves of constant width w , Reuleaux triangle has the least are . This is known as the Blaschke - Lebesque theorem . A recent proof of this result has been obtained by Evans Harrell .

Note that the Reuleaux triangle is not a C^1 regular curve for it has sharp corners . To obtain a C^1 example of a curve of constant width , we may take a curve which is a constant distance away from the Reuleaux triangle . Further , a C^ example may be constructed by taking an evolute of a deltoid .

1.4 BASICS OF EUCLIDEAN GEOMETRY

By R we shall always mean the set of real numbers . The set of all n - tuples of real numbers $R^n := \{ (p^3 , . . . , p^n) \mid p^i$ E R} is called the Euclidean n - space . So we have

p e $R^n \wedge p = (p^1 , . . . , p^n) , p^i$ *E* R .

Let p and q be a pair of points (or vectors) in R^n . We define p + q := ($p^1 + q^1$, . . . , $p^n + q^n$) . Further , for any scalar r E R , we define rp := (rp^1 , . . . , rp^n) . It is easy to show that the operations of addition and scalar multiplication that we have defined turn R^n into a vector space over the field of real numbers . Next we define the standard inner product on R^n by

$$(p, q) = p^1q^1 + \ldots + p^n q^n .$$

Note that the mapping (■ , ■) : R^n x R^n ^ R is linear in each variable and is symmetric . The standard inner product induces a norm on R^n defined by

IIpII := (p, p)2 .

If p E R , we usually write |p| instead of ||p|| .

The first nontrivial fact in Euclidean geometry , is the following important result which had numerous applications:

Theorem . (The Cauchy - Schwartz inequality) *For all* p *and* q in R^n |(p, q)| ^ *IIpH* llqH -

The equality holds if and only if p = Aq for some A E R .

Since this is such a remarkable and far reaching result we will include here three different proofs . The first proof is quite short and slick , but also highly nontransparent , i . e . , it is not easy to see how someone could come up with that . The second proof is perhaps more reasonable , but also more advanced . The third proof is the most elementary , but then again it is quite tricky .

Proof I (Quadratic Formula) . If p = Aq it is clear that equality holds . Oth - erwise , let f (A) := (p — Aq , p — Aq) . Then f (A) > 0 . Further , note that f (A) may be written as a quadratic equation in A:

$$f(A) = ||p||^2 - 2A(p, q) + A^2$$

Hence its discriminant must be negative:

$$4(p, q)^2 - 4||p||^2||q||^2 < 0$$

which completes the proof .

Proof II (Lagrange Multipliers) . Again suppose that p = Aq . Then

$(p, q) = ||p|| ||M|| (H$

Thus it suffices to prove that for all unit vectors p and q we have

$|(p, q)| < 1$

and equality holds if and only if p = ±q . This may be proved by using the method of lagrangne multipliers to find the maximum of the function (x , y) subject to the constraints ||x|| = 1 and ||y|| = 1 . More explicitly we need to find the critical points of

$$f(x, y, A_i, A_2) := (x, y) + A_i (||x||^2 - 1) + A_2 (||y||^2 - 1)$$

$$= y \sum_{i=1}^{n} (x_i V_i + A_1 x^2 + A_2 y^2) - A_1 - A_2 .$$

At a critical point we must have $0 = 5f / 5x_i = y_i + 2A_1 x_i$, which yields that V = ±x .

Proof (Induction) . First note that the case n = 1 is trivial . For n = 2 , the proof amounts to showing that

$$(p_1 q_1 + p_2 q_2)^2 < (p_1^2 + p_2^2)(q_1^2 + q_2^2) .$$

This is also easily verified by the expansion and simplification of both sides which reduces the above inequality to $(p_1 q_2 - q_2 p_1)^2 > 0$. Now suppose that the inequality we like to prove holds for n . Then to prove this for n + 1 note that The first inequlity above is just the inductive step , i . e . , the assumption that the inequality we want to prove holds for n , and the second inequality above is just an application of the case n = 2 which we established earlier . □

There is yet another proof of the Cauchy - Schwartz ineqaulity which com - bines ideas from the first and second proofs mentioned above , but avoids using either the quadratic formula or the Lagrange multipliers:

Exercise . (The simplest proof of the Cauchy - Schwartz

Notes

inequality)

Assume , as in the second proof above , that $||p|| = 1 = ||q||$ and note that

$$0 < ||p - (p \cdot q) q||^2 = (p - (p \cdot q) q \cdot p - q) q$$

whenever $p = Aq$. Expanding the right hand side yields the desired result . The standard Euclidean distance in R^n is given by

$$dist(p, q) := ||p - q|| .$$

An immediate application of the Cauchy - Schwartz inequality

Exercise . (The triangle inequality) Show that

dist (p , q) + dist (q , r) ^ dist (p , r)

for all p , q in R^n .

By a metric on a set X we mean a mapping d: X x X ^ R such that

1. d (p , q) ^ 0 , with equality if and only if p = q .
2. *d (p , q)* = d (q , p) .
3. d (p , q) + d (q , r) ^ d (p , r) .

These properties are called , respectively , positive - definiteness , symmetry , and the triangle inequality . The pair (X , d) is called a metric space . Using the above exercise , one immediately checks that (R^n , dist) is a metric space . Geometry , in its broadest definition , is the study of metric spaces , and Euclidean Geometry , in the modern sense , is the study of the metric space (R^n , dist) . Finally , we define the angle between a pair of nonzero vectors in R^n by

$$angle(p, q) := \cos^{-1} rrf^r \bullet$$

Note that the above is well defined by the Cauchy - Schwartz inequality . Now we have all the necessary tools to prove the most famous result in all of mathematics:

Exercise . (The Pythagorean theorem) Show that in a right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the length of the sides .

Next let us define the angle defined by 3 points p , o , q as

Zpoq := angle (p - o , q — o) = cos - „<p — 0'? — o}„ •

IP — o| |q — o|

Exercise . Suppose that p , o , q lie on a line and o lies between p and q . Show that then Zpoq = n .

We say that a set L C R^n is a line if there exists a pair of points p , q *E* R^n such that

L = {p + t (p — q) | t *E* R }•

A pair of lines in R^2 are called parallel if they do not intersect:

Exercise . (Playfair's Axiom) Show that , in R^2 , through every point p outside a line L there passes one and only one line parallel to L .

The Playfair's Axiom is equivalen to

Exercise . (Euclid's Fifth Postulate) Suppose that two parallel lines L , L in R^2 are intersected by a third line L" . Then the acute angles formed by these three lines are equal .

The next exercise is concerned with another corner stone of Euclidean Geometry , which may be proved using the last three exercises:

Exercise . (Sum of the angles in a triangle) Show that the sum of

the angles in a triangle is n (Hint: through one of the vertices draw a line parallel to the opposite side) .

The most important result in classical differential geometry is the Gauss - Bonnet theorem , which generalizes the fact proved in the last exercises to regions which lie on a curved surface .

Check your Progress 1

Discuss Differential Geometry

Discuss Curves

1.5 LET US SUM UP

In this unit we have discussed the definition and example of Differential Geometry … Cartography And Differential Geometry, Curves , Basics Of Euclidean Geometry

1.6 KEYWORDS

Differential Geometry … Cartography And Differential Geometry….. a cartographer and many terms in modern differential geometry (chart , atlas , map , coordinate system , geodesic

Curves ….. A (parametrized) curve (in Euclidean space) is a mapping a: I *u* R^n

Basics Of Euclidean Geometry …. By R we shall always mean the set of real numbers . The set of all n - tuples of real numbers R^n := { (p^4 , . . . , p^n) | p^i E R}

1.7 QUESTIONS FOR REVIEW

Explain Differential Geometry

Explain Cartography And Differential Geometry

Explain Curves

1.8 REFERENCE

Differential Geometry, Differential Geometry & Application,

Introduction to Defferential Geometry, Basic of Differential Geometry.

1.9 ANSWERS TO CHECK YOUR PROGRESS

Differential Geometry (Answer for Check your Progress - 1 Q)

Cartography And Differential Geometry (Answer for Check your Progress - 1 Q)

Curves (Answer for Check your Progress - 1 Q)

UNIT-II: DIFFERENTIAL GEOMETRY AND DIFFERENTIAL TOPOLOGY

STRUCTURE

2.0 Objectives

2.1 Introduction

2.2 Differential Geometry And Differential Topology

2.3 Tangent Spaces And Derivatives

2.4 The Inverse Function Theorem

2.5 Let Us Sum Up

2.6 Keywords

2.7 Questions For Review

2.8 Answers To Check Your Progress

2.9 Reference

2.0 OBJECTIVES

After studying this unit , you should be able to:

- Understand about Differential Geometry And Differential Topology
- Tangent Spaces And Derivatives
- The Inverse Function Theorem

2.1 INTRODUCTION

Differential geometry arose and developed as a result of and in connection to the mathematical analysis of curves and surfaces Mathematical analysis of curves and surfaces had been developed to answer some of unanswered questions that appeared in calculus like the reasons for relationships between complex shapes and curves , series and analytic functions Differential Geometry And Differential Topology , Tangent Spaces And Derivatives , The Inverse Function Theorem

2.2 DIFFERENTIAL GEOMETRY AND DIFFERENTIAL TOPOLOGY

Foundations

This chapter introduces various fundamental concepts that are central to the fields of differential geometry and differential topology . Both fields concern the study of smooth manifolds and their diffeomorphisms . The chapter begins with an introduction to submanifolds of Euclidean space and smooth maps to tangent spaces and derivatives and to submanifolds and embeddings . In we move on to vector fields and flows and introduce the Lie bracket of two vector fields . Lie groups and their Lie algebras , in the extrinsic setting , are the subject introduce vector bundles over a manifold as subbundles on a trivial bundle we prove the theorem of Frobenius . The last two sections of this chapter are concerned with carrying over all these concepts from the extrinsic to the intrinsic setting

.

Submanifolds of Euclidean Space

To carry out the Master Plan we must extend the definition of smooth map to maps f : X ^ Y between subsets X c Rk and Y c R which are not necessarily open . In this case a map f : X ^ Y is called smooth if for each x0 <= X there exists an open neighborhood U c Rk of xo and a smooth map F : U ^ R that agrees with f on U n X . A map f : X ^ Y is called a diffeomorphism if f is bijective and f and f - 1 are smooth . When there exists a diffeomorphism f : X ^ Y then X and Y are called diffeomorphic . When X and Y are open these definitions coin - cide with the usage in

Exercise (Chain Rule) . Let X C Rk , Y C R^ , Z C Rm be arbitrary subsets . If f : X ^ Y and g : Y ^ Z are smooth maps then so is the composition g o f : X ^ Z . The identity map id : X ^ X is smooth .

Notes

Exercise . Let E C Rk be an m - dimensional linear subspace and let vi , . . . , vm be a basis of E . Then the map f : Rm ^ E defined by f (x) := mi x - i - Vi is a diffeomorphism .

Definition . Let k , m <= No . A subset M C Rk is called a smooth m - dimensional submanifold of Rk iff every point p □ M has an open neighborhood U C Rk such that U n M is diffeomorphic to an open sub - set Q C Rm . A diffeomorphism

f : U n M ^ Q

is called a coordinate chart of M and its inverse

f := f - i : Q ^ U n M

is called a (smooth) parametrization of U n M .

Figure A coordinate chart f : U n M ^ Q .

In Definition we have used the fact that the domain of a smooth map can be an arbitrary subset of Euclidean space and need not be open . The term m - manifold in Rk is short for m - dimensional sub - manifold of Rk . In keeping with the Master Plan we will sometimes say manifold rather than submanifold of Rk to indicate that the context holds in both the intrinsic and extrinsic settings .

Lemma If M C Rk is a nonempty smooth m - manifold then m < k .

Proof . Fix an element p0 □ M , choose a coordinate chart f : U n M ^ Q with p0 □ U and values in an open subset Q C Rm , and denote its in - verse by f := f - i :Q ^ U n M . Shrinking U , if necessary , we may as - sume that f extends to a smooth map T : U ^ Rm . This extension satis - fies T (f (x)) = f (f (x)) = x and hence dt (f (x)) df (x) = id : Rm ^ Rm for all x □ Q , by the chain rule . Hence the derivative df (x) : Rm ^

Rk is in - jective for all x <= Q , and hence m < k because Q is nonempty . □

Example . Consider the 2 - sphere

M := S2 = { (x , y , z) e R3 | x2 + y2 + z2 = 1} depicted and let U C R3 and Q C R2 be the open sets

U := { (x , y , z) e R3 | z > 0} , Q := { (x , y) e R2 | x2 + y2 < 1} . The map 0 : U fl M ^ Q given by

0 (x , y , z) := (x , y) is bijective and its inverse ^ := 0 - 1 : Q ^ U f M is given by

^ (x , y) = (x , y , \ / 1 - x2 - y2) .

Since both 0 and ^ are smooth , the map 0 is a coordinate chart on S2 . Similarly , we can use the open sets z < 0 , y > 0 , y < 0 , x > 0 , x < 0 to cover S2 by six coordinate charts . Hence S2 is a manifold . A similar argument shows that the unit sphere Sm C Rm + 1 is a manifold for every integer m > 0 .

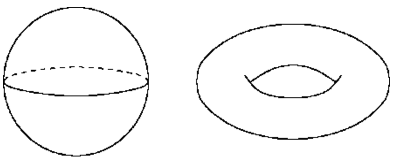


Figure : The 2 - sphere and the 2 - torus .

Example . Let Q C Rm be an open set and h : Q ^ Rk m be a smooth map . Then the graph of h is a smooth submanifold of Rm x Rk - m = Rk:

M = graph (h) := { (x , y) | x e Q , y = h (x) } .

It can be covered by a single coordinate chart 0 : U f M ^ V where U := Q x Rk - m , 0 is the projection onto Q , and ^ := 0 - 1 : Q ^ U is given by ^ (x) = (x , h (x)) for x e Q .

Exercise (The case m = 0) . Show that a subset M C Rk is a 0 - dimensional submanifold if and only if M is discrete , i . e . for every p e M there is an open set U C Rk such that U f M = {p} .

Exercise (The case m = k) . Show that a subset M C Rm is an m - dimensional submanifold if and only if M is open .

Exercisea (Products) . If Mi C Rki is an mi - manifold for i = 1 , 2

show that Mi x M2 is an (mi + m2) - dimensional submanifold of Rkl + k2 . Prove by induction that the n - torus Tn is a smooth submanifold of Cn .

The next theorem characterizes smooth submanifolds of Euclidean space . In particular condition (iii) will be useful in many cases for verifying the manifold condition . We emphasize that the sets U0 := U n M that appear in Definition are open subsets of M with respect to the relative topology that M inherits from the ambient space Rk and that such relatively open sets are also called M - open

Theorem (Manifolds) . Let m and k be integers with 0 < m < k . Let M C Rk be a set and p0 E M . Then the following are equivalent .

There exists an M - open neighborhood U0 C M of p0 and a diffeomor - phism

fo : Uo ^ Q0

onto an open set Q0 C Rm .

There exist open sets U , Q C Rk and a diffeomorphism f : U ^ Q such that p0 E U and

f (U n m) = Q n (Rm X {0}) .

There exists an open set U C Rk and a smooth map f : U ^ Rk - m such that p0 E U , the differential df (p) : Rk ^ Rk - m is surjective for every p E U n M , and

U n M = f - 1 (0) = {q E U | f (q) =0} .

Moreover , if (i) holds then the diffeomorphism f : U ^ Q in (ii) can be chosen such that U n M C U0 and f (p) = (f0 (p) , 0) for every p E U n M .

Proof . First assume (ii) and denote the diffeomorphism in (ii) by f = (fi , f2 , . . . , fk) : U ^ Q C Rk .

Then part (i) holds with U0 := U n M , Q0 := {x E Rm | (x , 0) E Q} , and

f0 := (f 1 , . . . , fm) |u0 : U0 ^ Q0 ,

and part (iii) holds with f := (fm + 1 , . . . , fk) : U ^ Rk - m . This shows that part (ii) implies both (i) and (iii) .

We prove that (i) implies (ii) . Let 0o : Uo ^ Qo be the coordinate chart in part (i) , let ^0 := 0 - 1 : Qo ^ U0 be its inverse , and denote

xo := 0o (po) e Qo .

Then , by Lemma the derivative d^o (xo) : Rm ^ Rk is an injective linear map . Hence there exists a matrix B e Rkx (k - m) such that

det ([d^o (xo) B]) = 0 .

Define the map ^ : Qo x Rk - m ^ Rk by

■0 (x , y) := ^o (x) + By .

Then the k x k - matrix d^ (xo , 0) = [d^o (xo) B] e Rkxk is nonsingular , by choice of B . Hence , by the Inverse Function Theorem , there exists an open neighborhood Q C Qo x Rk - m of (xo , 0) such that U := ^ (Q) C Rk is open and ^|^ : U ^ U is a diffeomorphism . In particular , the restriction of ^ to Q is injective . Now the set

Uo := |^o (x) | x e Q o , (x , 0) e Q j = |p e Uo | (0o (p) , 0) e Q j c M

is M - open and contains po . Hence , by the definition of the relative topology , there exists an open set W C Rk such that Uo = W n M . Define

U := U n W , Q:=Q n ^ - 1 (W) .

Then U n M = Uo and ^ restricts to a diffeomorphism from Q to U . Now let (x , y) e . We claim that

■0 (x , y) e M y = 0 .

Notes

If y = 0 then obviously ^ (x , y) = ^o (x) e M . Conversely , let (x , y) e Q and suppose that p := ^ (x , y) e M . Then p e U n M = U n W n M = Uo C Uo and hence (0o (p) , 0) e Q , by definition of Uo . This implies

^ (0o (p) , 0) = ^o (0o (p)) = p = ^ (x , y) .

Since the pairs (x^y) and (0o (p) , 0) both belong to the set Q and the re - striction of ^ to U is injective we obtain x = 0o (p) and y = 0 . It follows from (2 . 1 . 1) that the map 0 := (^|q) - 1 : U ^ Q satisfies 0 (U n M) = { (x , y) e Q | ^ (x , y) e M} = Q n (Rm x {0}) . Thus we have proved that (i) implies (ii) .

We prove that (iii) implies (ii) . Let f : U ^ Rk - m be as in part (iii) . Then p0 □ U and the derivative df (p0) : Rk ^ Rk - m is a surjective linear map . Hence there exists a matrix A <= Rmxk

Then det (d0 (po)) = 0 . Hence , by the Inverse Function Theorem , there exists an open neighborhood U' C U of p0 such that Q' := 0 (U') is an open subset of Rk and the restriction

0' := 0|w : U' ^ Q'

is a diffeomorphism . In particular , the restriction 0|w< is injective . Moreover , it follows fromn the assumptions on f and the definition of 0 that

U' n M = {p <= U'\ f (p) =0} = {p <= U'\ 0 (p) <= Rm x {0}}

and hence

0' (U' n M) = Q' n (Rm x {0}) .

Hence the diffeomorphism 0' : U' ^ Q' satisfies the requirements of part (ii) . This proves Theorem . □

Definition . Let U C Rk be an open set and f : U ^ R^ be a smooth function . An element c <= R^ is called a regular value of f if , for all p <= U , we have

f (p) = c =^ df (p) : Rk ^ R^ is surjective .

Otherwise c is called a singular value of f . Theorem asserts that , if c is a regular value of f the preimage

M := f - 1 (c) = {p <= U | f (p) = c}

is a smooth (k — <=) - dimensional submanifold of Rk .

2.3 TANGENT SPACES AND DERIVATIVES

The main reason for first discussing the extrinsic notion of embedded manifolds in Euclidean space as explained in the Master Plan is that the concept of a tangent vector is much easier to digest in the embedded case: it is simply the derivative of a curve in M , understood as a vector in the ambient Euclidean space in which M is embedded .

Tangent Space

Definition Let M c Rk be a smooth m - dimensional manifold and fix a point p e M . A vector v e Rk is called a tangent vector of M at p if there exists a smooth curve y : R ^ M such that

7 (0) = p , 7 (0) = v .

The set

TpM := {Y (0) | 7 : R ^ M is smooth , 7 (0) = p}

of tangent vectors of M at p is called the tangent space of M at p .

Theorem below shows that TpM is a linear subspace of Rk . As does any linear subspace it contains the origin; it need not actually intersect M . Its translate p + TpM touches M at p; this is what you should visualize for TpM .

Remark Let p e M be as in Definition and let v e Rk . Then

T M J > 0 : (- <= , <=) ^ M such that

V e p \ 7 is smooth , 7 (0) = p , y (0) = v .

Notes

To see this suppose that y : (—<= , <=) ^ M is a smooth curve with y (0) = p and Y (0) = v . Define 7 : R ^ M by

7 (t) := y (/ <=t ^ , t e R .

Y ( ) Y\V<=2 + fV ,

Then 7 is smooth and satisfies 7 (0) = p and "7 (0) = v . Hence v e TpM .

Theorem (Tangent spaces) . Let M C Rk be a smooth m - dimen - sional manifold and fix a point p e M . Then the following holds .

(i) Let U0 C M be an M - open set with p e U0 and let f0 : U0 ^ Q0 be a diffeomorphism onto an open subset Q0 C Rm . Let x0 := f0 (p) and let f0 := f - 1 : Q0 ^ U0 be the inverse map . Then

TPM = im (#0 (x0) : Rm ^ Rk)

Let U , Q C Rk be open sets and f : U ^ Q be a diffeomorphism such that p e U and f (U n M) = Q n (Rm x {0}) . Then

TPM = df (p) - 1 (Rm x {0}) .

Let U C Rk be an open neighborhood of p and f : U ^ Rk - m be a smooth map such that 0 is a regular value of f and U n M = f - 1 (0) . Then

TpM = ker df (p) .

TpM is an m - dimensional linear subspace of Rk .

Proof . Let f0 : Q0 ^ U0 and x0 e Q0 be as in (i) and let f : U ^ Q be as in (ii) . We prove that

imdf0 (x0) C TpM C df (p) - 1 (Rm x {0}) .

To prove the first inclusion choose a constant r > 0 such that

Br (x0) := {x e Rm | |x — x0| < r} C Q0 .

Now let f e Rm and choose e > 0 so small that

e |f| < r .

Then x0 + tf e Q0 for all t e R with |t| < e . Define 7 : (—e , e) ^ M by

Y (t) := f0 (x0 + tf) for — e < t < e .

Then 7 is a smooth curve in M satisfying

^0 (x0 + tf) = df 0 (x0) f .

To prove the second inclusion in we fix a vector v <= TpM . Then , by definition of the tangent space , there exists a smooth curve 7 : R ^ M such that y (0) = p and y (0) = v . Let U C Rk be as in (ii) and choose e > 0 so small that y (t) <= U for |t| < e . Then

0 (Y (t)) □ 0 (U n M) C Rm x {0}

for |t| < e and hence

d

(p) v = # (y (0)) 7 (0) = da

0 (Y (t)) <= Rm x {0} .

t=0

This shows that v <= d^ (p) - 1 (Rm x {0}) and thus we have proved .

Now the sets im d^0 (x0) and d^ (p) - 1 (Rm x {0}) are both m - dimensional linear subspaces of Rk . Hence it follows from that these subspaces agree and that they both agree with TpM . Thus we have proved asser - tions (i) , (ii) , and (iv) .

We prove (iii) . If v <= TpM then there is a smooth curve y : R ^ M such that y (0) = p and 7 (0) = v . For t sufficiently small we have y (t) <= U , where U C Rk is the open set in (iii) , and f (y (t)) = 0 . Hence

d

df (p) v = df (y (0)) y (0) = djt

and this implies TpM C ker df (p) . Since TpM and the kernel of df (p) are both m - dimensional linear subspaces of Rk we deduce that TpM = ker df (p) . □

Example . Let A = AT <= Rkxk be a nonzero matrix as in Example and let c = 0 . Then , by Theorem (iii) , the tangent space of the manifold

M = Ix <= Rk | xTAx = c} at a point x <= M is the k — 1 - dimensional linear subspace

TxM = {<= <= Rk | xTA<= = 0} .

Example . As a special case with A = 1 and c = 1 we find that the tangent space of the unit sphere Sm C Rm + 1 at a point x <= Sm is the orthogonal complement of x:

TxSm = x± = {<= <= Rm + 11 (x , <=) =0} .

Here (x , <=) = ^m=0 xi<=i denotes the standard inner product on Rm + 1 .

Exercise . What is the tangent space of the 5 - manifold

M := { (x , y) € R3 x R3 | \x — y\ = r} at a point (x , y) € M?

Example . Let H (x , y) := 2 \y\2 + V (x) be as in and let c be a regular value of H . If (x , y) € M := H - 1 (c) Then

T (x , y) M = { (<= , n) € Rn x Rn \ <y , n) + (VV (x) , <=) = 0} .

Here VV := (dV / dx1 , . . . , dV / dxn) : Rn ^ Rn denotes the gradient of V .

Exercise . The tangent space of SL (n , R) at the identity matrix is the

space { }

sl (n , R) := TiSL (n , R) = {<= € Rnxn \ trace (<=) = 0}

of traceless matrices .

Example The tangent space of O (n) at g is

TgO (n) = {v € Rnxn \ gTv + vTg = 0} .

In particular , the tangent space of O (n) at the identity matrix is the space of skew - symmetric matrices

o (n) := TiO (n) = {<= € Rnxn \ <=T + <= = 0}

To see this , choose a smooth curve R ^ O (n) : t ^ g (t) . Then g (t) Tg (t) = 1 for all t € R and , differentiating this identity with respect to t , we ob - tain g (t) Tg (t) + g (t) Tg (t) = 0 for every t . Hence every matrix v € TgO (n) satisfies the equation gTv + vTg = 0 . With this understood , the claim fol - lows from the fact that gTv + vTg = 0 if and only if the matrix <= := g - 1v is skew - symmetric and that the space of skew - symmetric matrices in Rnxn has dimension n (n — 1) / 2 .

Exercise . Let Q C Rm be an open set and h : Q ^ Rk - m be a smooth map . Prove that the tangent space of the graph of h at a point (x , h (x)) is the graph of the differential dh (x) : Rm ^ Rk - m:

M = { (x , h (x)) \ x € Q} , T (x , h (x)) M = { (<= dh (x) <=) \ <= € Rm} .

Exercise (Monge coordinates) . Let M be a smooth m - manifold in Rk and suppose that p e M is such that the projection TpM ^ Rm x {0} is invertible . Prove that there exists an open set Q C Rm and a smooth map h : Q ^ Rk - m such that the graph of h is an M - open neighborhood of p . Of course , the projection TpM ^ Rm x {0} need not be invertible , but it must be invertible for at least one of the (m) choices of the m dimensional coordinate plane . Hence every point of M has an M - open neighborhood which may be expressed as a graph of a function of some of the coordinates in terms of the others

Derivative

A key purpose behind the concept of a smooth manifold is to carry over the notion of a smooth map and its derivatives from the realm of first year analysis to the present geometric setting . Here is the basic definition . It appeals to the notion of a smooth map between arbitrary subsets of Euclidean spaces

Definition . Let M C Rk be an m - dimensional smooth manifold and

f : M ^ R*

be a smooth map . The derivative of f at a point p e M is the map

Notes

df (p) : TpM ^ R*

defined as follows . Given a tangent vector v e TpM choose a smooth curve

Y : R ^ M

satisfying

Now define the vector by

Y (0) = p , 7 (0) = v . df (p) v e R*

f (yM) = lim f (Y (h) { - f (p) .

t=0 h^° h

That the limit on the right in equation exists follows from our assumptions . We must prove , however , that the derivative is well defined , i . e . that the right hand side of depends only on the tangent vector v and not on the choice of the curve 7 used in the definition . This is the content of the first assertion in the next theorem .

Theorem (Derivatives) . Let M C Rk be an m - dimensional smooth manifold and f : M ^ R be a smooth map . Fix a point p e M . Then the following holds .

The right hand side of is independent of 7 .

The map df (p) : TpM ^ R is linear .

If N C R is a smooth n - manifold and f (M) C N then

df (p) TpM C Tf (p) N .

(Chain Rule) Let N be as in (iii) , suppose that f (M) C N , and let g : N ^ Rd be a smooth map . Then

d (g o f) (p) = dg (f (p)) o df (p) : TpM ^ Rd .

If f = id : M ^ M then df (p) = id : TpM ^ TpM .

Proof . We prove (i) . Let v e TpM and 7 : R ^ M By definition there is an open neighborhood U C Rk of p and a smooth map F : U ^ R^ such that

F (p') = f (p') for all p e U n M .

Let dF (p) e R^xk denote the Jacobian matrix (i . e . the matrix of all first partial derivatives) of F at p . Then , since 7 (t) e U n M for t sufficiently small , we have

dF (p) v = dF (7 (0)) 7 (0) d dt d dt

The right hand side of this identity is independent of the choice of F while the left hand side is independent of the choice of 7 . Hence the right hand side is also independent of the choice of 7 and this proves (i) . Assertion (ii) follows immediately from the identity

df (p) v = dF (p) v

just established .

Assertion (iii) follows directly from the definitions . Namely , if 7 is as in Definition then

P := f o 7 : R ^ N is a smooth curve in N satisfying

P (0) = f (7 (0)) = f (p) =: q , (3 (0) = df (p) v =: w .

Hence w <= TqN . Assertion (iv) also follows directly from the definitions . If g : N ^ Rd is a smooth map and P , q , w are as above then

d dt

g (P (t))

t=0

= dg (q) w = dg (f (p)) df (p) v .

and this proves (iv) . Assertion (v) follows directly from the definitions and this proves Theorem . □

Corollary (Diffeomorphisms) . Let M C Rk be a smooth m - mani - fold and N C R^ be a smooth n - manifold and let f : M ^ N be a diffeomor -

phism . Then m = n and the differential df (p) : TpM ^ Tf (p) N is a vector space isomorphism with inverse

df (p) - 1 = df - l (f (p)) : Tf (p) N ^ TpM

for all p <= M .

Proof . Define g := f - 1 : N ^ M so that

g o f = idM , f o g = idN .

Then it follows from Theorem that , for p <= M and q := f (p) <= N , we have dg (q) o df (p) = id : TPM ^ TPM , df (p) o dg (q) = id : TqN ^ TqN . Hence df (p) : TpM ^ TqN is a vector space isomorphism with inverse

dg (q) = df (p) - 1 : TqN ^ TpM .

Hence m = n and this proves Corollary .

2.4 THE INVERSE FUNCTION THEOREM

Corollary is analogous to the corresponding assertion for smooth maps between open subsets of Euclidean space . Likewise , the inverse function theorem for manifolds is a partial converse of Corollary

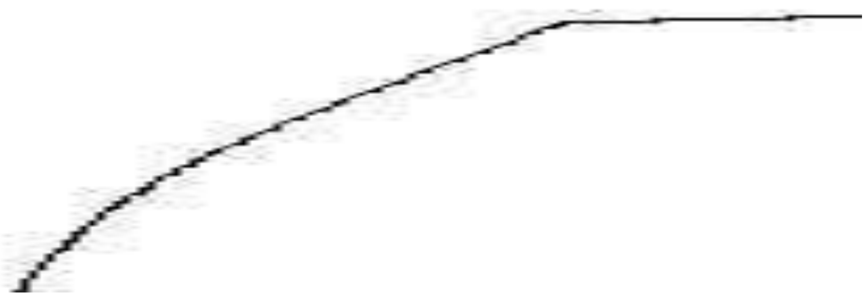


Figure : The Inverse Function Theorem .

Theorem (Inverse Function Theorem) . Assume that M C Rk and N C R are smooth n - manifolds and f : M ^ N is a smooth map . Let p0 E M and suppose that the differential df (p0) : Tp0M ^ Tf (P0) N is a vector space isomorphism . Then there is an M - open neighborhood U C M of p0 such that V := f (U) C N is an N - open subset of N and the restric - tion f | u : U ^ V is a diffeomorphism .

Proof . Choose coordinate charts <fi0 : U0 ^ U0 , defined on an M - open neighborhood U0 C M of p0 onto an open set U0 C Rn , and ^0 : V0 ^ Vfj^defined on an N - open neighborhood V0 C N of q0 := f (p0) onto an open set V0 C Rn . Shrinking U0 , if necessary , we may assume that f (U0) C V0 . Then the map

f := ^0 o f o 0 - 1 : U0 ^ V0

is smooth and its differential df (x0) : Rn ^ Rn is bijective at x0 := 00 (p0) . Hence the Invers^ Function Theorem asserts that there exists an open neighborhood U C IJp ofx0 such that f := f (U) is an open subset of VU0 and the restriction of fU to UU is a diffeomorphism from UU to U . Hence the assertion holds with U := ^ - 1 (f /) and V := ^ - 1 (U) . This proves Theorem □

Definition (Regular value) . Let M C Rk be a smooth m - manifold , let N C R^ be a smooth n - manifold , and let f : M ^ N be a smooth map . An element q E N is called a regular value of f if , for every p E M with f (p) = q , the differential df (p) : TpM ^ Tf (p) N is surjective .

Theorem (Regular Values) . Let f : M ^ N be as in Definition and let q e N be a regular value of f . Then the set

P := f - 1 (q) = {p e M | f (p) = q}

is a smooth submanifold of Rk of dimension m — n and , for each point p e P , its tangent space at p is given by

TpP = ker df (p) = {v e TVM | df (p) v = 0} .

Proof . Let p0 e P and choose a coordinate chart : U0 ^ fo (Uo) C Rm on an M - open neighborhood U0 C M of p0 . Likewise , choose a coordi - nate chart : V0 ^ ^o (Vo) C Rn on an N - open neighborhood V0 C N of q . Shrinking U0 , if necessary , we may assume that f (U0) C V0 . Then the point c0 := ^0 (q) is a regular value of the map

Notes

f0 := ^0 o f o 0 - 1 : ^ (Ud) ^ Rn .

Namely , if x e <f0 (U0) satisfies f0 (x) = c0 , then p := 0 - 1 (x) e U0 n P , so the maps df - 1 (x) : Rm ^ TPM , df (p) : TPM ^ TqN , and d^ (q) : TqN ^ Rn are all surjective , hence so is their composition , and by the chain rule this composition is the derivative df0 (x) : Rm ^ Rn . With this understood , it follows from Theorem that the set

f0_1 (c0) = {x e MU) | f (0 - 1 (x)) = q} = ^ (U n P)

is a manifold of of dimension m — n contained in the open set f0 (U0) C Rm . Using Definition and shrinking U0 further , if necessary , we may as - sume that the set f0 (U0 n P) is diffeomorphic to an open subset of Rm - n . Compsing this diffeomorphism with <f0 we find that U0 C P is diffeomorphic to the same open subset of Rm - n . Since the set U0 C M is M - open , there exists an open set U C Rk such that U n M = U0 , hence U n P = U0 n P , and so U0 n P is a P - open neighborhood of p0 . Thus we have proved that every element p0 e P has a P - open neigborhood that is diffeomorphic to an open subset of Rm - n . Thus P C Rk is a manifold of dimension m — n

Now let p e P and v e TpP . Then there exists a smooth curve 7 : R ^ P such that 7 (0) = p and 7 (0) = v . Since f (7 (f)) = 0 for all t , we have

df (p) v = df

f (Y (t)) = 0

t=0

and so v e ker df (p) . Hence TpP C ker df (p) and equality holds because both TpP and ker df (p) are (m — n) - dimensional linear subspaces of Rk . This proves Theorem .

Submanifolds and Embeddings

This section deals with subsets of a manifold M that are themselves mani - folds as in Definition . Such subsets are called submanifolds of M .

Definition (Submanifold) . Let M C Rk be an m - dimensional man - ifold . A subset L C M is called a submanifold of M of dimension I , if L itself is an <= - manifold .

Definition (Embedding) . Let M C Rk be an m - dimensional mani - fold and N C R be an n - dimensional manifold . A smooth map f : N ^ M is called an immersion if its differential df (q) : TqN ^ Tf (q) M is infective for every q e N . It is called proper if , for every compact subset K C f (N) , the preimage f - 1 (K) = {q e N | f (q) e K} is compact . The map f is called an embedding if it is a proper infective immersion .

Remark In our definition of proper maps it is important that the compact set K is required to be contained in the image of f . The literature also contains a stronger definition of proper which requires that f - 1 (K) is a compact subset of M for every compact subset K C N , whether or not K is contained in the image of f . This holds if and only if the map f is proper in the sense of Definition and has an M - closed image .

Figure : A coordinate chart adapted to a submanifold .

Theorem (Submanifolds) . Let M C Rk be an m - dimensional man - ifold and N C R^ be an n - dimensional manifold .

If f : N ^ M is an embedding then f (N) is a submanifold of M .

If P C M is a submanifold then the inclusion P ^ M is an embedding .

A subset P C M is a submanifold of dimension n if and only if , for every p0 e P , there exists a coordinate chart f : U ^ Rm on an M - open neighborhood U of p0 such that f (U n P) = f (U) n (Rn x {0}) .

A subset P C M is a submanifold of dimension n if and only if , for ev - ery p0 e P , there exists an M - open neighborhood U C M of p0 and a smooth map g : U ^ Rm - n such that 0 is a regular value of g and U n P = g - 1 (0) .

Notes

Lemma (Embeddings) . Let M and N be as in Theorem let f : N ^ M be an embedding , let q0 <= N , and define

P := f (N) , po := f (qo) <= P .

Then there exists an M - open neighborhood U C M of p0 , an N - open neighborhood V C N of q0 , an open neighborhood W C Rm - n of the origin , and a diffeomorphism F : V x W ^ U such that , for all q <= V and all z <= W ,

F (q , 0) = f (q) ,

F (q , z) <= P z = 0 .

Proof . Choose any coordinate chart fa : U0 ^ Rm on an M - open neighbor - hood Uo C M of p0 . Then d (fo ◦ f) (q0) = #0 (f (q0)) ◦ df (q0) : TqoN ^ Rm is injective . Hence there is a linear map B : Rm - n ^ Rm such that the map

Tq0N x Rm - n ^ Rm : (w , ( ) ^ d (fa ◦ f) (q0) w + B

is a vector space isomorphism . Define the set

Q := { (q , z) <= N x Rm - n | f (q) <= U0 , fa (f (q)) + Bz <= fa (U0) } .

This is an open subset of N x Rm - n and we define F : Q ^ M by

F (q , z) := 0 - 1 (00 (f (q)) + Bz) .

This map is smooth , it satisfies F (q , 0) = f (q) for all q f - 1 (U0) , and the derivative dF (q0 , 0) : Tq0 N x Rm - n ^ Tp0M is the composition of the map with dfa (p0) - 1 : Rm ^ Tp0M and so is a vector space isomor - phism . Thus the Inverse Function Theorem asserts that there is an N - open neighborhood V0 C N of q0 and an open neighborhood W0 C Rm - n of the origin such that V0 x W0 C Q , the set U0 := F (V0 x W0) is M - open , and the restriction of F to V0 x W0 is a diffeomorphism onto U0 . Thus we have constructed a diffeomorphism F : V0 x W0 ^ U0

We claim that the restriction of F to the product V x W of sufficiently small open neighborhoods V C N of q0 and W C Rm - n of the origin also satisfies . Otherwise , there exist sequences qi <= V0 converging to q0 and zi <= W0 \ {0} converging to zero such that F (qi , zi) <= P . Hence there exists a sequence qi <= N such that F (qi , zi) = f (qi) . This sequence converges to f (q0) . Since f is proper we may assume , passing to a suitable subsequence if necessary , that qii converges to a point q0i <= N . Then

f (q0) = lim f (qi) = lim F (qi , zi) = f (q0) .

i^^ i^^

Since f is injective , this implies q'0 = q0 . Hence (qi , 0) e V0 x W0 for i sufficiently large and F (qi , 0) = f (qi) = F (qi , Zi) . This contradicts the fact that the map F : Vo x Wo ^ M is injective

Proof of Theorem We prove (i) . Let q0 e N , denote p0 := f (q0) e P , and choose a diffeomorphism F : V x W ^ U as in Lemma . Then set V C N is diffeomorphic to an open subset of Rn (after schrinking V if necessry) , the set U n P is P - open because U C M is M - open , and we have U n P = {F (q , 0) | q e V} = f (V) Hence the map f : V ^ U n P is a diffeomorphism whose inverse is the composition of the smooth maps F - 1 : U n P ^ V x W and V x W ^ V : (q , z) ^ q . Hence a P - open neighborhood of p0 is diffeomorphic to an open subset of Rn . Since p0 e P was chosen arbitrary , this shows that P is an n - dimensional submanifold of M .

We prove (ii) . The inclusion i : P ^ M is obviously smooth and in - jective (it extends to the identity map on Rk) . Moreover , TpP C TpM for every p e P and the differential di (p) : TpP ^ TpM is the obvious inclusion for every p e P . That i is proper follows immediately from the definition . Hence i is an embedding .

We prove (iii) . If a coordinate chart f0 as in (iii) exists then the set U0 n P is P - open and is diffeomorphic to an open subset of Rn . Since the point p0 e P was chosen arbitrary this proves that P is an n - dimensional submanifold of M . Conversely , suppose that P is an n - dimensional sub - manifold of M and let p0 e P . Choose any coordinate chart f0 : U0 ^ Rm of M defined on an M - open neighborhood U0 C M of p0 . Then f0 (U0 n P) is an n - dimensional submanifold of Rm . Hence Theorem asserts that there are open sets V , W C Rm with p0 e V C f0 (U0) and a diffeo - morphism ^ : V ^ W such that fo (po) e V , ^ (V n fo (Uo n P)) = W n (Rn x {0}) .

Now define U := f - ^V) C U0 . Then p0 e U , the chart f0 restricts to a diffeomorphism from U to V , the composition f := ^ o ^0|u : U ^ W is a diffeomorphism , and f (U n P) = ^ (V n f0 (U0 n P)) = W n (Rn x {0}) .

We prove (iv) . That the condition is sufficient follows directly from Theorem . To prove that it is necessary , assume that P C M , is a submanifold of dimension n , fix an element po e P , and choose a coordi - nate chart f : U ^ Rm on an M - open neighborhood U C M of p0 as in part (iii) . Define the map g : U ^ Rm - n by g (p) := (fn + 1 (p) , . . . , fm (p)) for p e U . Then 0 is a regular value of g and g - 1 (0) = U n P . This proves Theorem□

Example . Let S1 C R2 = C be the unit circle and consider the map f : S1 ^ R2 given by f (x , y) := (x , xy) . This map is a proper immersion but is not injective (the points (0 , 1) and (0 , - 1) have the same image under f) . The image f (S1) is a figure 8 in R2 and is not a submanifold .

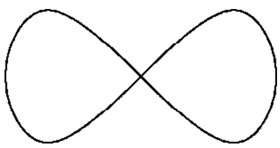


Figure : A proper immersion .

Example . Consider the restriction of the map f in to the submanifold N := S1 \ { (0 , - 1) } . The resulting map f : N ^ R2 is an injective immersion but it is not proper . It has the same image as before and hence f (N) is not a manifold .

Example . The map f : R ^ R2 given by f (t) := (t2 , t3) is proper and injective , but is not an embedding (its differential at x = t is not injective) . The image of f is the set f (R) = C := { (x , y) e R2 | x3 = y2} and is not a submanifold

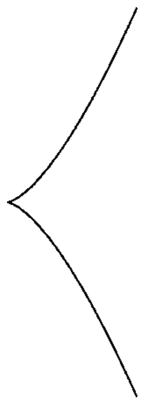


Figure: A proper injection .

Example . Define the map f : R ^ R2 by f (t) := (cos (t) , sin (t)) . This map is an immersion , but it is neither injective nor proper . However , its image is the unit circle in R2 and hence is a submanifold of R2 . The map R ^ R2 : t ^ f (t3) is not an immersion and is neither injective nor proper , but its image is still the unit circle .

Check your Progress - 1

Discuss Differential Geometry

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Discuss Differential Topology

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2.5 LET US SUM UP

In this unit we have discussed the definition and example of Differential Geometry And Differential Topology , Tangent Spaces And Derivatives , The Inverse Function Theorem

2.6 KEYWORDS

Differential Geometry And Differential Topology ….. This chapter introduces various fundamental concepts that are central to the fields of differential geometry and differential topology

Tangent Spaces And Derivatives ….. The main reason for first discussing the extrinsic notion of embedded manifolds in Euclidean space

The Inverse Function Theorem….. Corollary is analogous to the corresponding assertion for smooth maps between open subsets of Euclidean space . Likewise , the inverse function theorem for manifolds is a partial converse of Corollary

2.7 QUESTIONS FOR REVIEW

Explain Differential Geometry

Explain Differential Topology

2.8 ANSWERS TO CHECK YOUR PROGRESS

Differential Geometry (answer for Check your Progress- 1 Q)
Differential Topology (answer for Check your Progress -1 Q)

2.9 REFERENCE

Differential Geometry, Differential Geometry & Application, Introduction to Defferential Geometry, Basic of Differential Geometry.

UNIT-III: VECTOR FIELDS AND FLOWS … VECTOR FIELDS

STRUCTURE

3.0 Objectives

3.1 Introduction

3.2 Vector Fields and Flows… Vector Fields

3.3 Lie Groups

3.4 Lie Group Homeomorphisms

3.5 Let Us Sum Up

3.6 Keywords

3.7 Questions For Review

3.8 Answers To Check Your Progress

3.9 References

3.0 OBJECTIVES

After studying this unit , you should be able to:

- Understand about Vector Fields and Flows… Vector Fields
- Lie Groups
- Lie Group Homeomorphisms

3.1 INTRODUCTION

Differential geometry arose and developed as a result of and in connection to the mathematical analysis of curves and surfaces Mathematical analysis of curves and surfaces had been developed to answer some of unanswered questions that appeared in calculus like the reasons for relationships between complex shapes and curves , series and analytic functions Vector Fields and Flows… Vector Fields , Lie Groups , Lie Group Homeomorphisms

3.2 VECTOR FIELDS AND FLOWS VECTOR FIELDS

Definition (Vector Field) . Let M C Rk be a smooth m - manifold . A (smooth) vector field on M is a smooth map X : M ^ Rk such that

X (p) e TpM

for every p e M . The set of smooth vector fields on M will be denoted by

Vect (M) := {X : M ^ Rk | X is smooth , X (p) e TpM for all p e m} .

Exercise Prove that the set of smooth vector fields on M is a real vector space .

Example . Denote the standard cross product on R3 by

/ X2V3 - X3V2 \ x x y := I X3P1 - Xiy3 I \ xiy2 - X2yi J

For x , y e R3 . Fix a vector <= e S2 and define the maps X , Y : S2 ^ R3 by

X (p) := <= x p , Y (p) := (<= x p) x p .

These are vector fields with zeros ±□ . Their integral curves

Example . Let M := R2 . A vector field on M is then any smooth map X : R2 ^ R2 . As an example consider the vector field

X (x , y) := (x , - y) .

This vector field has a single zero at the origin and its integral curves

Definition (Integral curves) . Let M C Rk be a smooth m - manifold , let X <= Vect (M) be a smooth vector field on M , and let I C R be an open interval . A smooth map 7 : I ^ M is called an integral curve of X if it satisfies the equation

Y (t) = X (y (t))

for every t <= I .

Theorem . Let M C Rk be a smooth m - manifold and X <= Vect (M) be a smooth vector field on M . Fix a point p0 <= M . Then the following holds .

(i) There is an open interval I C R containing 0 and a smooth curve Y : I ^ M satisfying the equation

At (t) = X (Y (t)) Y (0) = Po

for every t <= I .

(ii) If Yi : I1 ^ M and Y2 : I2 ^ M are two solutions of on open intervals I1 and I2 containing 0 , then y 1 (t) = Y2 (t) for every t <= I1 n I2 .

Proof . We prove (i) . Let f0 : U0 — Rm be a coordinate chart on M , defined on an M - open neighborhood U0 C M of p0 . The image of f0 is an open set

Q := fo (Uo) C Rm

and we denote the inverse map by - f0 : rem the differential d^0 (x) : Rm — tangent space Tp0 (x) M for every x e Q .

f (x) := d^0 (x) 1X (0o (x)) , x G Q .

This map is smooth and hence , by the basic existence and uniqueness the - orem for ordinary differential equations in Rm the equation

x (t) = f (x (t)) , x (0) = xo := fo (po) ,

has a solution x : I — Q on some open interval I C R containing 0 . Hence the function

Y := - f0 o x : I ^ U0 C M

is a smooth solution of . This proves (i) .

Notes

The local uniqueness theorem asserts that two solutions Yi : Ii ^ M for i = 1 , 2 agree on the interval (- <= , e) C I1 n I2 for e > 0 sufficiently small . This follows immediately from the standard uniqueness theorem for the solutions and the fact that x : I ^ Q is a solution of if and only if y := ◦ x : I ^ U0 is a solution . To prove (ii) we observe that the set

I := Ii n I2

is an open interval containing zero and hence is connected . Now consider the set

A := {t e 11 Yi (t) = Y2 (t) } .

This set is nonempty , because 0 G A . It is closed , relative to I , because the maps Y1 : I ^ M and y2 : I ^ M are continuous . Namely , if ti G I is a sequence converging to t e I then Y1 (ti) = Y2 (ti) for every i and , taking the limit i —y to , we obtain Y1 (t) = Y2 (t) and hence t e A . The set A is also open by the local uniqueness theorem . Since I is connected it follows that A = I . □The Flow of a Vector Field

Definition (The flow of a vector field) . Let M C Rk be a smooth m - manifold and X e Vect (M) be a smooth vector field on M . For p0 e M the maximal existence interval of p0 is the open interval

I C R is an open interval containing 0 and there is a solution x : I ^ M

: I (p0) ^ M . The flow of X is the map $: D ^ M defined by

D := { (t , po) | P0 e M , t e I (Po) }

and $ (t , p0) := 7 (t) , where 7 : I (p0) ^ M is the unique solution .

Theorem . Let M C Rk be a smooth m - manifold and X e Vect (M) be a smooth vector field on M . Let $: D ^ M be the flow of X . Then the following holds .

D is an open subset of R x M .

The map $: D ^ M is smooth .

Let p0 e M and s e I (p0) . Then

I ($ (s , p0)) = I (p0) - s

and , for every t e R with s + 1 e I (p0) , we have

$ (s + t , p0) = $ (t , $ (s , p0)) .

Lemma . Let M , X , D , $ be as in Theorem and let K C M be a compact set . Then there exists an M - open set U C M and an e > 0 such that K C U , (—e , e) x U CD , and $ is smooth on (—e , e) x U .

Proof . In the case where M = Q is an open subset for every p e M , there exists an M - open neighborhood Up C M of p and an ep > 0 such that (—ep , ep) xUp CD and the restriction of $ to (—ep , ep) x Up is smooth .

Using this observation for every p e K

(and the axiom of choice) we obtain an M - open cover K C (JpeK Up . Since the set K is compact there exists a finite subcover K C Upi U ■ ■ ■ U UpN =: U . Now define e := min{epi , . . . , epN } to deduce that (— e , e) x U CD and $ is smooth on (—e , e) x U .

Proof of Theorem . We prove (iii) . The map 7 : I (p0) — s ^ M defined by Y (t) := f (s + t , p0) is a solution of the initial value problem 7 (t) = X (7 (t)) with 7 (0) = f (s , p0) . Hence I (p0) — s C I (f (s , p0)) and equation holds for every t <= R with s + t <= I (p0) . In particular , with t = —s , we have p0 = f (—s , f (s , p0)) . Thus we obtain equality in equation by the same argument with the pair (s . p0) replaced by (—s , f (s , p0)) .

We prove (i) and (ii) . Let (t0 , p0) <= D so that p0 <= M and t0 <= I (p0) . Suppose t0 > 0 . Then K := {f (t , p0) | 0 < t < t0} is a compact subset of M . (It is the image of the compact interval [0 , t0] under the unique solution 7 : I (p0) ^ M .) Hence , by Lemma , there is an M - open set U C M and an e > 0 such that

K C U , (—e , e) x U CD ,

and f is smooth on (—e , e) x U . Choose N so large that t0 / N < e . Define U0 := U and , for k = 1 , . . . , N , define the sets Uk C M inductively byUfc := {p <= U | f (t0 / N , p) <= Ufc - i} .

These sets are open in the relative topology of M .

We prove by induction on k that (—e , kt0 / N + e) x Uk CD and f is smooth on (—e , kt0 / N + e) x Uk . For k = 0 this holds by definition of e and U . If k <= {1 , . . . , N} and the assertion holds for k — 1 then we have

p <= Ufc   p <= U , f (t0 / N , p) <= Ufc - i

(—e , e) C I (p) , (—e , (k — 1) t0 / N + e) C I (f (t0 / N , p)) (—e , kt0 / N + e) C I (p) .

Here the last implication follows Moreover , for p <= Ufc and t0 / N — e < t < kt0 / N + e , we have , by (2 . 4 . 4) , that

f (t , p) = f (t — t0 / N , f (t0 / N , p))

Since f (t0 / N , p) <= Uk - 1 for p <= Uk the right hand side is a smooth map on the open set (t0 / N — e , kt0 / N + e) x Uk . Since Uk C U , f is also a smooth map on (—e , e) x Uk and hence on (—e , kt0 / N + e) x Uk . This completes the induction . With k = N we have found an open neighborhood of (t0 , p0) contained in D , namely the set (—e , t0 + e) x UN , on which f is smooth . The case t0 < 0 is treated similarly . This proves (i) and (ii)

Definition . A vector field X <= Vect (M) is called complete if , for each p0 <= M , there is an integral curve 7 : R ^ M of X with 7 (0) = p0 .

Lemma Let M c Rk is a compact manifold . Then every vector field on M is complete .

Proof . Let X e Vect (M) . It follows from Lemma with K = M that there exists an e > 0 such that (- <= , e) c I (p) for all p e M .

By Theorem this implies I (p) = R for all p e M . Hence X is complete . □

Let M c Rk be a smooth manifold and X e Vect (M) . Then

X is complete I (p) = R V p e M   D = R x M .

Assume X is complete , let 0 : R x M ^ M be the flow of X , and define the map 0t : M ^ M by 4>t (p) := 0 (t , p) for t e R and p e M . Then Theorem 2 . 4 . 9 asserts that 0t is smooth for every t e R and that

0s + t = 0s o 0t , 00 = id

for all s , t e R . In particular this implies that 0t o 0 - t = 0 - t o 0t = id . Hence 0t is bijective and (0t) - 1 = 0 - t , so each 0t is a diffeomorphism .

Exercise . Let M c Rk be a smooth manifold . A vector field X on M is said to have compact support if there exists a compact subset K c M such that X (p) = 0 for every p e M \ K . Prove that every vector field with compact support is complete .

We close this subsection with an important observation about incomplete vector fields . The lemma asserts that an integral curve on a finite existence interval must leave every compact subset of M .

Lemma . Let M c Rk be a smooth m - manifold , let X e Vect (M) , let 0 : D ^ M be the flow of X , let K c M be a compact set , and let p0 e M be an element such that

I (p0) n [0 , to) = [0 , b) , 0 < b < to .

Then there exists a number 0 < tK < b such that

tK <t<b =^ 0 (t , po) e M \ K

Proof . By Lemma there exists an e > 0 such that (- e , e) c I (p) for every p e K . Choose e so small that e < b and define

tK := b — e > 0 .

Choose a real number tK < t < b . Then I (0 (t , po)) = [0 , b — t) by equation in part (ii) of Theorem Since 0 <b — t < b — tk = e , this shows that (—e , e) c I (0 (t , po)) and hence 0 (t , po) e K .

Corollary Let M c Rk be a smooth m - manifold , let X e Vect (M) , and let 7 : (0 , T) ^ M be an integral curve of X . If there exists a compact set K c M that contains the image of 7 , then 7 extends to an integral curve of X on the interval (- p , T + p) for some p> 0 .

Proof . Here is another more direct proof that does not rely on Lemma

Since K is compact , there exists a constant c > 0 such that |X (p) | < c for all p e K . Since 7 (t) e K for 0 <t <T , this implies

< |7 (r) | dr = |X (q (r)) | dr < c (t — s)

for 0 < s < t < T . Thus the limit p0 := limt^0 Y (t) exists in Rk and , since K is a closed subset of Rk , we have p0 e K c M . Define y0 : [0 , T) ^ M by

p0 , for t = 0 ,

Y (t) , for 0 < t < T .

We prove that yo is differentiable at t = 0 and Y0 (0) = X (p0) . To see this , fix a constant e > 0 . Since the curve [0 , T) ^ Rk : t ^ X (y (t)) is continuous , there exists a constant 5 > 0 such that

|X (Y (t)) — X (Po) | < e .

0 < t < 5 =^ Hence , for 0 < s < t < 5 , we have

|Y (t) — Y (s) — (t — s) X (Po) | =

(7 (r) — X (po)) dr

(X (y (r)) — X (po)) dr

s

f |X (Y (r)) — X (po) | dr J s

(t — s) e .

Take the limit s —> 0 to obtain

lim |Y (t) — Y (s) — (t — s) X (Po) | < e s^0 t — s _

for 0 < t < 5 . Thus y0 is differentiable at t = 0 with Y0 (0) = X (p0) , as claimed . Hence y extends to an integral curve 7 : (—p , T) ^ M of X for some p > 0 via 7 (t) := 0 (t , p0) for —p < t < 0 and 7 (t) := Y (t) for 0 < t < T . Here 7 is the flow of X . That y also extends beyond t = T , follows by replacing Y (t) with y (T—t) and X with —X .

The Group of Diffeomorphisms

Let us denote the space of diffeomorphisms of M by

Diff (M) := (0 : M ^ M | 0 is a diffeomorphism} .

This is a group . The group operation is composition and the neutral element is the identity . Now equation asserts that the flow of a complete vector field X e Vect (M) is a group homomorphism

R ^ Diff (M) : t ^ 00

This homomorphism is smooth and is characterized by the equation

d

j / (p) = X (0t (p)) , 0o (p) = p

for all p e M and t e R . We will often abbreviate this equation in the form

d

— 0t = X o 0t , 00 = id . dt

Exercise (Isotopy) . Let M C Rk be a compact manifold and I C R be an open interval containing 0 . Let

I x M ^ Rk : (t , p) ^ Xt (p)

be a smooth map such that Xt e Vect (M) for every t . Prove that there is a smooth family of diffeomorphisms I x M ^ M : (t , p) ^ 0t (p) satisfying for every t e I . Such a family of diffeomorphisms

I ^ Diff (M) : t ^ 0t

is called an isotopy of M . Conversely prove that every smooth isotopy I ^ Diff (M) : t ^ 0t is generated (uniquely) by a smooth family of vector fields I ^ Vect (M) : t ^ Xt . 2 . 4 . 3 The Lie Bracket

Let M c Rk and N C R be smooth m - manifolds and X G Vect (M) be smooth vector field on M . If 4 : N ^ M is a diffeomorphism , the pullback of X under 4 is the vector field on N defined by

(4*X) (q) := # (q) - 1X (4 (q))

for q G N . If 0 : M ^ N is a diffeomorphism then the pushforward of X under 4 is the vector field on N defined by

(0*X) (q) := d0 (0 - 1 (q)) X (0 - 1 (q)) (2 . T9)

for q G N .

Lemma . Let M c Rk , N c R , and P C Rn be smooth m - dimen - sional submanifolds and let X G Vect (M) and Z G Vect (P) . Then

0*X = (0 - 1) *X

and

(4 o 0) *X = 4*0*X , (4 o 0) *Z = 0*4) *Z .

Proof . Equation follows from the fact that

d0 - 1 (q) = d0 (0 - 1 (q)) - 1 : TqN ^ T^ - i{q) M

for all q G N

We think of a vector field on M as a smooth map

X : M ^ Rkthat satisfies the condition X (p) G TpM for every p G M . Ignoring this condition temporarily , we can differentiate X as a map from M to Rk and its differential at p is then a linear map

dX (p) : TpM ^ Rk .

In general , this differential will no longer take values in the tangent space TpM . However , if we have two vector fields X and Y on M the next lemma shows that the difference of the derivative of X in the direction Y and of Y in the direction X does take values in the tangent spaces of M .

Definition (Lie Bracket) . Let M C Rk be a smooth manifold and let X , Y <= Vect (M) be smooth vector fields on M . The Lie bracket of X and Y is the vector field [X , Y] <= Vect (M) defined by

[X , Y] (p) := dX (p) Y (p) - dY (p) X (p) .

Lemma . Let M C Rk and N C R be smooth manifolds , let X , Y , Z be smooth vector fields on M , and let

f : N ^ M

be a diffeomorphism . Then

f * [X , Y] = [f*X , f*Y] , [X , Y] + [Y , X] =0 ,

[X , [Y , Z]] + [Y , [Z , X]] + [Z , [X , Y]] = 0 .

The last equation is called the Jacobi identity

Proof . Let R ^ Diff (M) : t ^ f' be the flow of Y . Then the map

R ^ Diff (N) : t ^ f 1 o o f

is the flow of the vector field f*Y on N .

Definition . A Lie algebra is a real vector space g equipped with a skew - symmetric bilinear map g x g ^ g : (C , rj) ^ [C , n] that satisfies the Jacobi identity [C , [n , Z]] + [n , [Z , C]] + [Z , [C , n]] = 0 for all C , n , Z G g -

Example . The Vector fields on a smooth manifold M C Rk form a Lie algebra with the Lie bracket . The space gl (n , R) = Rnxn of real n x n - matrices is a Lie algebra with the Lie bracket

[C n] := Cn - nC .

It is also interesting to consider subspaces of gl (n , R) that are invariant under this Lie bracket . An example is the space

T

o (n) := {C G gl (n , R) | CT + C = 0}

Notes

of skew - symmetric n x n - matrices . It is a nontrivial fact that every finite - dimensional Lie algebra is isomorphic to a Lie subalgebra of gl (n , R) for some n . For example , the cross product defines a Lie algebra STRUCTURE on R3 and the resulting Lie algebra is isomorphic to o (3) .

Remark . There is a linear map Rmxm ^ Vect (Rm) : C ^ Xg which assigns to a matrix C G gl (m , R) the linear vector field Xg : Rm ^ Rm given by Xg (x) := Cx for x G Rm . This map preserves the Lie bracket , i . e . [Xg , Xn] = X[g , n] , and hence is a Lie algebra homomorphism .

Example . The configuration space for driving a car in the plane is the manifold M := C x S1 , where S1 C C denotes the unit circle . Thus a point in M is a pair p = (z , A) e C x C with | A| = 1 . The point z e C represents the position of the car and the unit vector A e S1 represents the direction in which it is pointing . The left turn is represented by a vector field X and the right turn by a vector field Y on M . These vector field are given by

X (z , A) := (A , iA) , Y (z , A) := (A , - iA) .

Their Lie bracket is the vector field

[X , Y] (z , A) = (—2iA , 0) .

This vector field represents a sideways move of the car to the right . And a sideways move by 2e2 can be achieved by following a backward right turn for time e , then a backward left turn for time e , then a forward right turn for time e , and finally a forward left turn for time e .

This example can be reformulated by identifying C with R2 via z = x + iy and representing a point in the unit circle by the angle 9 e R / 2nZ via A = eld . In this formulation the manifold is M = R2 x R / 2nZ , a point in M is represented by a triple (x , y , 9) e R3 , the vector fields X and Y are

X (x , y , 9) := (cos (9) , sin (9) , 1) , Y (x , y , 9) := (cos (9) , sin (9) , —1) , and their Lie bracket is [X , Y] (x , y , 9) = 2 (sin (9) , — cos (9) , 0) .

Lemma . Let X , Y e Vect (M) be complete vector fields on a man - ifold M and 00 04 e Diff (M) be the flows of X and Y , respectively . Then the Lie bracket [X , Y] vanishes if and only if the flows of X and Y commute , i . e . 0s o 0t = 0t o 0s for all s , t e R .

Proof . If the flows of X and Y commute then the Lie bracket [X , Y] vanishes by Lemma . Conversely , suppose that [X , Y] = 0 . Then we have

d d

Ts (0s) , Y = (0s) , - (0r) , Y = (0s) , [X , Y] = 0 for every s e R and hence

(0s) , Y = Y .

Fix a real number s and define the curve 7 : R ^ M by y (t) := 0s (0t (p)) for t e R . Then 7 (0) = 0s (p) and

7 (t) = d0s (0t (p)) Y (0t (p)) = ((0s) , Y) (Y (t)) = Y (Y (t))

for all t . Here the last equation follows Since 0t is the flow of Y we obtain q (t) = 0t (0s (p)) for all t e R

3.3 LIE GROUPS

Combining the concept of a group and a manifold , it is interesting to consider groups which are also manifolds and have the property that the group operation and the inverse define smooth maps . We shall only consider groups of matrices .

Definition and Examples

Definition (Lie Group) . A nonempty subset G C Rnxn is called a Lie group if it is a submanifold of Rnxn and a subgroup of GL (n , R) , i . e .

g , h e G =^ gh e G

Notes

(where gh denotes the product of the matrices g and h) and

g e G =^ det (g) = 0 and g - 1 e G .

(Since G = 0 it follows from these conditions that the identity matrix 1 is an element of g .)

Example . The general linear group G = GL (n , R) is an open subset of Rnxn and hence is a Lie group the special linear group

SL (n , R) = {g e GL (n , R) | det (g) = 1}

is a Lie group and , Special orthogonal group

SO (n) := {g e GL (n , R) | gTg = 1 , det (g) = 1}

is a Lie group . In fact every orthogonal matrix has determinant ±1 and so SO (n) is an open subset of O (n) (in the relative topology) .

In a similar vein the group GL (n , C) := {g e Cnxn | det (g) = 0} of com - plex matrices with nonzero (complex) determinant is an open subset of Cnxn and hence is a Lie group . As in the real case , the subgroups

SL (n , C) := {g e GL (n , C) | det (g) = 1} ,

U (n) := {g e GL (n , C) | g*g = 1} ,

SU (n) := {g e GL (n , C) | g*g = il , det (g) = 1}

are submanifolds of GL (n , C) and hence are Lie groups . Here g* := gT denotes the conjugate transpose of a complex matrix .

Exercise . Prove that SL (n , C) , U (n) , and SU (n) are Lie groups . Prove that SO (n) is connected and that O (n) has two connected components .

Exercise . Prove that GL (n , C) can be identified with the group

G := {$ e GL (2n , R) | $Jq = Jq$} , Jo := ^ 0 - J ^ .

Hint: Use the isomorphism Rn x Rn ^ Cn : (x , y) ^ x + iy . Show that a matrix $ e R2nx2n commutes with J0 if and only if it has the form

$ = (Y "x) , X , Y e Rnxn .

What is the relation between the real determinant of $ and the complex determinant of X + iY?

Exercise . Let J0 be and define

Sp (2n) := {t e GL (2n , R) | TTJoT = Jo} .

This is the symplectic linear group . Prove that Sp (2n) is a Lie group .

Example (Unit Quaternions) . The Quaternions form a four - dimensional associative unital algebra H , equipped with a basis 1 , i , j , k . The elements of H are vectors of the form

x = x0 + ixi + jx2 + kx3 x0 , xi , x2 , x3 e R .

The product STRUCTURE is the bilinear map H x H ^ H : (x , y) ^ xy , determined by the relations

i2 = j2 = k2 = - 1 , ij = - ji = k , jk = - kj = i , ki = - ik = j .

This product STRUCTURE is associative but not commutative . The quaternions are equipped with an involution H ^ H : x ^ x , which assigns to a quaternion x of the form its conjugate x := x0 - ix1 - jx2 - kx3 . This involution satisfies the conditions

x + y = x + y , xy = yx , xx = |x|2 , |xy| = |x| |y|

for x , y e H , where |x| := y / x2 + x2 + x2 + x2 denotes the Euclidean norm of the quaternion . Thus the unit quaternions form a group

Sp (1) := {x e H | |x| = 1}

with the inverse map x ^ x . Note that the group Sp (1) is diffeomorphic to the 3 - sphere S3 C R4 under the isomorphism H = R4 .

Let G C GL (n , R) be a Lie group . Then the maps

G x G ^ G : (g , h) ^ gh , G ^ G : g ^ g - 1

are smooth (see [18]) . Fixing an element h <= G we find that the derivative of the map G ^ G : g ^ gh at g <= G is given by the linear map

TgG ^ TghG : g ^ gh .

Here g and h are both matrices in Rnxn and gh denotes the matrix prod - uct . In fact , if g <= TgG then , since G is a manifold , there exists a smooth curve y : R ^ G with y (0) = g and y (0) = g . Since G is a group we obtain a smooth curve ft : R ^ G given by ft (t) := Y (t) h . It satisfies ft (0) = gh and so gh = ft (0) <= Tghg .

The linear map is obviously a vector space isomorphism whose inverse is given by right multiplication with h - 1 . It is sometimes convenient to define the map Rh : G ^ G by

Rh (g) := gh

for g <= G (right multiplication by h) . This is a diffeomorphism and the linear map is the derivative of Rh at g , so

dRh (g) g = gh for g <= TgG .

Similarly , each element g <= G determines a diffeomorphism Lg : G ^ G , given by

Lg (h) := gh

for h <= G (left multiplication by g) . Its derivative at h <= G is again given by matrix multiplication , i . e . the linear map dLg (h) : ThG ^ Tghg is given by

dLg (h) h = gh for h <= ThG .

Since Lg is a diffeomorphism its differential dLg (h) : ThG ^ TghG is again a vector space isomorphism for every h <= G .

Exercise . Prove that the map G ^ G: g ^ g - 1 is a diffeomorphism and that its derivative at g <= G is the vector space isomorphism

TgG ^ Tg - 1G : v ^ - g - 1 vg - 1 .

The Lie Algebra of a Lie Group

Let

G C GL (n , R)

be a Lie group . Its tangent space at the identity matrix 1 e G is called the Lie algebra of G and will be denoted by

g = Lie (G) := Ttg .

This terminology is justified by the fact that g is in fact a Lie algebra , i . e . it is invariant under the standard Lie bracket operation

[<= , n] := <=n - n<=

on the space Rnxn of square matrices The proof requires the notion of the exponential matrix . For <= e Rnxn and t e R we define

^ tk ck

exp (t<=) :=^ — .

k=0

A standard result in first year analysis asserts that this series converges absolutely (and uniformly on compact t - intervals) , that the map

r ^ Rnxn : t ^ exp (t<=) is smooth and satisfies the differential equation
d

— exp (t<=) = <= exp (t<=) = exp (t<=) <= ,

and that

exp ((s + 1) <=) = exp (s<=) exp (t<=) , exp (0<=) = H

for all s , t e R . This shows that the matrix exp (t<=) is invertible for each t and that the map R ^ GL (n , R) : t ^ exp (t<=) is a group homomorphism .

Exercise . Prove the following analogue of For <= , n e g

exp (Vt<=) exp (Vtn) exp (—Vt<=) exp (- Vtn) = [<= , n]

t=0

In other words , the infinitesimal Lie group commutator is the matrix commutator .

Lemma . Let G C GL (n , R) be a Lie group and denote by g := Lie (G) its Lie algebra . Then the following holds .

If <= <= g then exp (t<=) <= G for every t <= R .

If g <= G and n <= g then gpg 1 <= g .

If <= , n <= g then [<= , n] = <=n - n<= <= g .

Proof . We prove (i) . For every g <= G we have a vector space isomor - phism g = TIG ^ TgG : <= ^ <=g as in Hence each element <= <= g determines a vector field Xg <= Vect (G) , defined by

Xg (g) := <=g <= TgG , g <= G .

By Theorem there is an integral curve 7 : (- <= , <=) ^ G satisfying

7 (t) = Xg (Y (t)) = <=7 (t) , Y (0) = H .

By the curve (- <= , e) ^ Rnxn : t ^ exp (t<=) satisfies the same initial value problem and hence , by uniqueness , we have exp (t<=) = 7 (t) <= G for all t <= R with |t| < <= . Now let t <= R and choose N <= N such that |N < <= . Then exp (N<=) <= G and hence it follows from that

(t \N exp (t<=) =exp Nj<=) <= G .

This proves (i) .

We prove (ii) . Consider the smooth curve 7 : R ^ Rnxn defined by

Y (t) := g exp (tn) g - 1 .

By (i) we have 7 (t) <= G for every t <= R . Since 7 (0) = 1 we have

gng - 1 = y (o) <= g .

This proves (ii) .

We prove (iii) . Define the smooth map n : R ^ Rnxn by

n (t) := exp (t<=) n exp (- t<=) .

By (i) we have exp (t<=) <= G and , by (ii) , we have n (t) <= g for every t <= R . Hence [<= , n] = n (0) <= g . This proves (iii) and Lemma □

By Lemma the curve 7 : R ^ G defined by 7 (t) := exp (t<=) g is the integral curve of the vector field Xg in with initial condition 7 (0) = g . Thus Xg is complete for every <= <= g .

Lemma If f G g and y : R ^ G is a smooth curve satisfying

Y (s + t) = Y (s) Y (t) Y (0) = 1 Y (0) = f , (2 - 5 - 9) then y (t) = exp (tf) for every t G R .

Proof . For every t G R we have

Hence y is the integral curve of the vector field Xg in with y (0) = 1 . This implies y (t) = exp (tf) for every t G R , as claimed . □

Example . Since the general linear group GL (n , R) is an open subset of Rnxn its Lie algebra is the space of all real n x n - matrices

gl (n , R) := Lie (GL (n , R)) = Rnxn .

The Lie algebra of the special linear group is

s! (n , R) := Lie (SL (n , R)) = {f G gl (n , R) | trace (f) = 0} and the Lie algebra of the special orthogonal group is

so (n) := Lie (SO (n)) = jf G gl (n , R) | fT + f = 0 j = o (n)

Exercise . Prove that the Lie algebras of the general linear group over C , the special linear group over C , the unitary group , and the special unitary group are given by

gl (n , C) := Lie (GL (n , C)) = Cnxn ,

s! (n , C) := Lie (SL (n , C)) = {f G gl (n , C) | trace (f) = 0} ,

u (n) := Lie (U (n)) = {f G gl (n , R) | f* + f = 0} ,

su (n) := Lie (SU (n)) = {f G gl (n , C) | f* + f = 0 , trace (f) = 0} .

These are vector spaces over the reals . Determine their real dimensions . Which of these are also complex vector spaces?

Exercise . Let G C GL (n , R) be a subgroup . Prove that G is a Lie group if and only if it is a closed subset of GL (n , R) in the relative topology .

3.4 LIE GROUP HOMOMORPHISMS

Let G and H be Lie groups and g and h be Lie algebras . A Lie group homomorphism from G to H is a smooth map p : G ^ H that is a group homomorphism . A Lie group isomorphism is a bijective Lie group homomorphism whose inverse is also a Lie group homomorphism . A Lie algebra homomorphism from g to h is a linear map that preserves the Lie bracket .

Lemma . Let G and H be Lie groups and denote their Lie algebras by g := Lie (G) and h := Lie (H) . Let p : G ^ H be a Lie group homomor - phism and denote its derivative at 1 e G by

p := dp (1) : g ^ h .

Then p is a Lie algebra homomorphism .

Proof . The proof has three steps .

Step 1 . For all f e g and t e R we have p (exp (tf)) = exp (tp (f)) .

Fix an element f e g . Then , by Lemma , we have exp (tf) e G for every t e R . Thus we can define a map 7 : R ^ H by 7 (t) := p (exp (tf)) . Since p is smooth , this is a smooth curve in H and , since p is a group ho - momorphism and the exponential map satisfies , our curve 7 satisfies the conditions

Y (s + t) = Y (s) Y (t) Y (0) = 1 Y (0) = dp (1) f = p (f) .

Hence it follows from Lemma 2 . 5 . 10 that y (t) = exp (tp ({)) . This proves Step 1 .

Step 2 . For all g e G and r e g we have p (grg - 1) = p (g) p (rj) p (g - 1) .

Define the smooth curve y : R ^ G by y (t) := g exp (tr) g - 1 . This curve takes values in G by Lemma 2 . 5 . 9 . By Step 1 we have

p (Y (t)) = p (g) p (exp (tr)) p (g) - 1 = p (g) exp (tp (r)) p (g) - 1

for every t . Since y (0) = 1 and - 7 (0) = grg - 1 we obtain p (grg - 1) = dp (Y (0)) ^ (0) d

p (Y (t))

t=0

p (g) exp (t / ^ (r)) p (g

t=0

= p (g) p5 (r) p (g 1) .

This proves Step 2 .

Step 3 . For all <= , n S g we have

/ K^nD = [p (0 , p> (n)] .

Define the curve n : R ^ g by

n (t) := exp (t{) n exp (—1{)

for t S R . By Lemma this curve takes values in the Lie algebra of G and

n (0) = [C , n] .

Hence

p (exp (t{) n exp (—1<=))

t=0

p (exp (tC)) p (n) p (exp (- tC))

t=0

exp MCW / ^ (n) exp (- t / ^ (C))

t=0

= [/ ^ (C) , / 5 (n)] .

Here the first equation follows from the fact that p is linear , the second equation follows from Step 2 with g = exp (t{) , and the third equation follows from Step 1 . This proves Step 3 and Lemma . □

Notes

Example The complex determinant defines a Lie group homomorphism det : U (n) ^ S1 . The associated Lie algebra homomorphism is

trace = det : u (n) ^ iR = Lie (S1) .

Example (Unit Quaternions and SU (2)) . The Lie group SU (2) is diffeomorphic to the 3 - sphere . Every matrix in SU (2) can be written as

g = (Xo + iXl X2 + iX3) , x0 + x1 + = 1 . (2 . 5 . 10) V —x2 + ix3 x0 - iX1 j 0 1 2 3 v y

Here the Xi are real numbers . They can be interpreted as the coordinates of a unit quaternion x = x0 + ix1 + jx2 + kx3 S Sp (1) . The reader may verify that the map Sp (1) ^ SU (2) : x ^ g in is a Lie group isomorphism .

Exercise (The double cover of SO (3)) . Identify the imaginary part of H with R3 and write a vector <= e R3 = Im (H) as a purely imagi - nary quaternion <= = i<=? + j<=? + k<=3 . Prove that if <= e Im (H) and x e Sp (1) then x<=x e Im (H) . Define the map p : Sp (1) ^ SO (3) by

p (x) <= := x<=X

for x e Sp (1) and <= e Im (H) . Prove that the linear map p (x) : R3 ^ R3 is represented by the 3 x 3 - matrix

(x2 + x? — x^ — x| 2 (x?x2 — x0x3) 2 (x1x3 + x0x2) \ 2 (x1x2 + x0x3) x0 + x2 — x2 — x? 2 (x2x3 — x0x?) I . 2 (x?x3 — x0x2) 2 (x2x3 + x0x?) x0 + x2 — x? — x2 J

Show that p is a Lie group homomorphism . Find a formula for the map

p := dp (il) : sp (1) ^ so (3)

and show that it is a Lie algebra isomorphism . For x , y e Sp (1) prove that p (x) = p (y) if and only if y = ±x .

Example . Consider the map

GL (n , R) ^ Diff (Rn) : g ^

which assigns to every nonsingular matrix g e GL (n , R) the linear diffeo - morphism : Rn ^ Rn given by (x) := gx for x e Rn . This map g ^ $g is a group homomorphism . The group Diff (Rn) is infinite dimensional and thus cannot be a Lie group . However , it has many properties in common with Lie groups . For example one can define what is meant by a smooth path in Diff (Rn) and extend formally the notion of a tangent vector (as the derivative of a path through a given element of Diff (Rn)) to this set - ting . In particular , the tangent space of Diff (Rn) at the identity can then be identified with the space of vector fields

TidDiff (Rn) = Vect (Rn) .

Differentiating the map g ^ 0g , one then obtains a linear map

gl (n , R) ^ Vect (Rn) : <= ^ Xg

which assigns to every matrix <= e g ! (n , R) the vector field Xg : Rn ^ Rn given by Xg (x) := <=x for x e Rn .

Example . Let g be a finite dimensional Lie algebra . Then the set . , , IN
T is a biiective linear map ,

Aut (g>:=|T: g"g

of Lie algebra automorphisms of g is a Lie group . Its Lie algebra is the space of derivations on g denoted by

A is a linear map ,

A[C , n] = [AC , n] + [C , An] v c , n e g

Now suppose that g = Lie (G) is the Lie algebra of a Lie group G . Then there is a map

ad : G ^ Aut (g) , ad (g) n := gng - 1 ,

for g e G and n e g . Lemma (ii) asserts that ad (g) is indeed a linear map from g to itself for every g e G . The reader may verify that the map

ad (g) : g ^ g

Notes

is a Lie algebra automorphism for every g e G and that the map ad : G ^ Aut (g) is a Lie group homomorphism . The associated Lie algebra homo - morphism is the map

Ad : g ^ Der (g) , Ad (C) n :=[C , n] ,

for C , n e g . To verify the claim Ad = ad we compute

exp (tc) n exp (- tc) = [C , n] .

t=o

Exercise . Let g be any Lie algebra and define the map

Ad : g ^ End (g) . Prove that the endomorphism

Ad (C) : g ^ g

is a derivation for every C e g and that Ad : g ^ Der (g) is a Lie algebra homomorphism . If g is finite dimensional , prove that Aut (g) is a Lie group with Lie algebra Der (g) .

Smooth Maps and Algebra Homomorphisms

Let M be a smooth submanifold of Rk . Denote by F (M) := C^ (M , R) the space of smooth real valued functions f : M ^ R . Then F (M) is a commutative unital algebra . Each p e M determines a unital algebra ho - momorphism ep : F (M) ^ R defined by ep (f) = f (p) for p e M .

Theorem . Every unital algebra homomorphism e : F (M) ^ R has the form e = ep for some p e M .

Proof . Assume that e : F (M) ^ R is an algebra homomorphism .

Claim . For all f , g e F (M) we have e (g) = 0 =^ e (f) e f (g - 1 (0)) . Indeed , the function f — e (f) ■ 1 lies in the kernel of e and so the func - tion h := (f — e (f) ■ 1) 2 + g2 also lies in the kernel of e . There must be at least one point p e M where h (p) = 0 for otherwise 1 = e (h) e (1 / h) = 0 . For this point p we have f (p) = e (p) and g (p) = 0 , hence p e g - 1 (0) , and therefore e (f) = f (p) e f (g - 1 (0)) . This proves the claim .

The theorem asserts that there exists a p e M such that every f e F (M) satisfies e (f) = f (p) . Assume , by contradiction , that this is false . Then for every p e M there exists a function f e F (M) such that f (p) = e (f) . Re - place f by f — e (f) to obtain f (p) = 0 = e (f) . Now use the axiom of choice to obtain a family of functions fp e F (M) , one for every p e M , such that fp (p) = 0 = e (fp) for all p e M . Then the set Up := f - 1 (R \ {0}) is an M - open neighborhood of p for every p e M . Choose a sequence of com - pact sets Kn C M such that Kn C intM (Kn + 1) for all n and M = IJn Kn . Then , for each n , there is a gn e F (M) (a finite sum of the form ^i fp .) such that e (gn) = 0 and gn (q) > 0 for all q e Kn . If M is compact , this is already a contradiction because a positive function cannot belong to the kernel of e . Otherwise , choose f e F (M) such that f (q) > n for all q e M \ Kn and all n e N . Then e (f) e f (g - 1 (0)) C f (M \ Kn) C [n , to) by the claim and so e (f) > n for all n . This is a contradiction

Now let N be another smooth submanifold (say of R^) and let C^ (M , N) denote the space of smooth maps from M to N . A homomorphism from F (N) to F (M) is a (real) linear map T : F (N) ^ F (M) that satisfies

T (fg) = T (f) T (g) , T (1) = 1 .

An automorphism of the algebra F (M) is a bijective homomorphism T : F (M) ^ F (M) . Let Hom (F (N) , F (M)) denote the space of ho - momorphisms from F (N) to F (M) . The automorphisms of F (M) form a group denoted by Aut (F (M)) .

Corollary . The pullback operation

C~ (M , N) ^ Hom (F (N) , F (M)) : 0 ^ 0*

is bijective . In particular , the map Diff (M) ^ Aut (F (M)) : 0 ^ 0* is an anti - isomorphism of groups .

Proof . This is an exercise with hint . Let T : F (N) ^ F (M) be a unital algebra homomorphism . There exists a map 0 : M ^ N

such that ep o T = e^ (p) for all p e M . Prove that f o 0 : M ^ R is smooth for every smooth map f : N ^ R and deduce that 0 is smooth .

Remark . The pullback operation is functorial , i . e .

(0 o 0) * = 0* o 0* , idM = idF (M) .

for 0 e C^ (M , N) and 0 e C^ (N , P) . Here id denotes the identity map of the space indicated in the subscript . Hence Corollary may be summarized by saying that the category of smooth manifolds and smooth maps is anti - isomorphic to a subcategory of the category of commutative unital algebras and unital algebra homomorphisms .

Exercise . If M is compact , then there is a slightly different way to prove Theorem An ideal in F (M) is a linear subspace J C F (M) satisfying the condition f e F (M) , g e J =^ fg e J . A maximal ideal in F (M) is an ideal J C F (M) such that every ideal J' C F (M) containing J is equal to J . Prove that , if M is compact and J C F (M) is an ideal with the property that for every p e M there is an f e J with f (p) = 0 , then J = F (M) . Deduce that each maximal ideal in F (M) has the form Jp := {f e F (M) | f (p) = 0} for some p e M .

Vector Fields and Derivations

A derivation of F (M) is a linear map 5 : F (M) ^ F (M) that satisfies

5 (fg) = 5 (f) g + / 5 (g) .

and the derivations form a Lie algebra denoted by Der (F (M)) . We may think of Der (F (M)) as the Lie algebra of Aut (F (M)) with the Lie bracket given by the commutator . the pullback operation

Diff (M) ^ Aut (F (M)) : 0 ^ 0*

can be thought of as a Lie group anti - isomorphism . Differentiating it at the identity 0 = id gives a linear map

Vect (M) ^ Der (F (M)) : X ^ Lx .

Here the operator LX : F (M) ^ F (M) is given by the derivative of a function f in the direction of the vector field X , i . e .

d

LXf := df • X = S

t=0

where 0t denotes the flow of X . Since the map is the derivative of the "Lie group" anti - homomorphism we expect it to be a Lie algebra anti - homomorphism . Indeed , one can show that

l[x , y] = lylx — LXly = - [LX , ly]

for X , Y <= Vect (M) . This confirms that our sign in the definition of the Lie bracket is consistent with the standard conventions in the theory of Lie groups . In the literature the difference between a vector field and the associated derivation LX is sometimes neglected in the notation and many authors write Xf := df • X = LXf , thus thinking of a vector field on a manifold M as an operator on the space of functions . With this notation one obtains the equation [X , Y]f = Y (Xf) — X (Yf) and here lies the origin for the use of the opposite sign for the Lie bracket in many books on differential geometry .

Check your Progress – 1

Discuss Vector Fields and Flows

Discuss Lie Groups

3.5 LET US SUM UP

In this unit we have discussed the definition and example of Vector Fields and Flows… Vector Fields , Lie Groups , Lie Group Homeomorphisms

3.6 KEYWORDS

Vector Fields and Flows… Vector Fields ….. Let M C Rk be a smooth m - manifold A (smooth) vector field on M is a smooth map X : M ^ Rk

Lie Groups….. Combining the concept of a group and a manifold , it is interesting to consider groups which are also manifolds and have the property that the group operation and the inverse define smooth maps

Lie Group Homeomorphisms Let G and H be Lie groups and g and h be Lie algebras . A Lie group homomorphism from G to H is a smooth map p : G ^ H that is a group homomorphism

3.7 QUESTIONS FOR REVIEW

Explain Vector Fields and Flows

Explain Lie Groups

3.8 ANSWERS TO CHECK YOUR PROGRESS

Vector Fields and Flows (answer for Check your Progress - 1 Q)

Lie Groups (answer for Check your Progress - 1 Q)

3.9 REFERENCE

Differential Geometry, Differential Geometry & Application, Introduction to Defferential Geometry, Basic of Differential Geometry.

UNIT-IV : VECTOR BUNDLES AND SUBMERSIONS

STRUCTURE

4.0 Objectives

4.1 Introduction

4.2 Vector Bundles And Submersions

4.3 The Implicit Function Theore

4.4 Let Us Sum Up

4.5 Keywords

4.6 Questions For Review

4.7 Answers To Check Your Progress

4.8 References

4.0 OBJECTIVES

After studying this unit , you should be able to:

- Understand about Vector Bundles And Submersions
- The Implicit Function Theore

4.1 INTRODUCTION

Differential geometry arose and developed as a result of and in connection to the mathematical analysis of curves and surfaces Mathematical analysis of curves and surfaces had been developed to answer some of unanswered questions that appeared in calculus like the reasons for relationships between complex shapes and curves , series and analytic functions Vector Bundles And Submersions , The Implicit Function Theore

Notes

4.2 VECTOR BUNDLES AND SUBMERSIONS

Submersions

Let M c Rk be a smooth m - manifold and N C R be a smooth n - manifold . A smooth map f : N ^ M is called a submersion if its derivative

df (q) : TqN ^ Tf {q) M

is surjective for every q e N .

I

g f M U P0

Figure : A local right inverse of a submersion .

Lemma . Let M c Rk be a smooth m - manifold , N c R be a smooth n - manifold , and f : N ^ M be a smooth map . The following are equivalent .

f is a submersion .

For every q0 e N there is an M - open neighborhood U of p0 := f (q0) and a smooth map g : U ^ N such that g (f (q0)) = q0 and f o g = id : U ^ U . Thus f has a local right inverse near every point in N (see Figure 2 . 10) .

Proof . We prove that (i) implies (ii) . Since the derivative

df (qo) : TqoN ^ TPoM

is surjective we have n > m and

dimker df (q0) = n — m .

Hence there is a linear map A : R ^ Rn - m whose restriction to the kernel of df (q0) is bijective . Now define the map ^ : N ^ M x Rn - m by

^ (q) := (f (q) , A (q— qo))

for q e N . Then ^ (q0) = (p0 , 0) and its derivative

df> (qo) : Tqo N ^ TPo M x Rn - m

sends w e Tq0N to (df (q0) w , Aw) and is therefore bijective . Hence it follows from the inverse function theorem for manifolds that there is an N - open neighborhood V C N of q0 such that the set

W := ^ (N) C M x Rn - m

is an open neighborhood of (p0 , 0) and ^\y : V ^ W is a diffeomorphism . Let

U := {p e M \ (p , 0) e W} and define the map g : U ^ N by

g (p) := ^ - 1 (p , 0) .

Then p0 e U , g is smooth and

0) = ^ (g (p)) = (f (g (p)) , A (g (p) - qo)) .

Hence f (g (p)) = p for all p e U and

g (po) = ^ - 1 (po , 0) = qo .

Corollary . The image of a submersion f : N ^ M is open .

Proof . If p0 = f (q0) e f (N) then the neighborhood U C M of p0 in Lemma (ii) is contained in the image of f . □

Corollary . If N is a nonempty compact manifold , M is a connected manifold , and f : N ^ M is a submersion then f is surjective and M is compact .

Proof . The image f (M) is an open subset of M by Corollary , it is a relatively closed subset of M because N is compact , and it is nonempty because N is nonempty . Since M is connected this implies that f (N) = M . In particular , M is compact . □

Notes

Exercise . Let f : N ^ M be a smooth map . Prove that the sets {q e N \ df (q) is injective} and {q e N \ df (q) is surjective} are open (in the relative topology of N) .

Vector Bundles

Let M C Rk be an m - dimensional smooth manifold . A (smooth) vector bundle (over M of rank n) is a smooth submanifold E C M x R of dimension m + n such that , for every p <= M , the set

is an n - dimensional linear subspace of R (called the fiber of E over p) . If E C M x R is a vector bundle then a (smooth) section of E is smooth map s : M ^ R such that s (p) <= Ep for every p <= M . A vector bundle E over M is equipped with a smooth map

n : E —> M

defined by n (p , v) := p This map is called the canonical projection of E . A section s : M ^ R of E determines a smooth map a : M ^ E which sends the point p <= M to the pair (p , s (p)) <= E . This map satisfies

n o a = id .

It is sometimes convenient to abuse notation and eliminate the distinction between s and a . Thus we will sometimes use the same letter s for the map from M to R and the map from M to E .

Example . Let M C Rk be a smooth m - dimensional submanifold .

The set

TM := { (p , v) | p <= M , v <= TpM}

is called the tangent bundle of M . This is a subset of M x Rk and , for every p <= M , its fiber TpM is an m - dimensional linear subspace of Rk by Theorem . However , it is not immediately clear from the definition that TM is a submanifold of M x Rk . The sections of TM are the vector fields on M .

Exercise . Let f : M ^ N be a smooth map between manifolds . Prove that the tangent map

TM ^ TN : (p , v) ^ (f (p) , df (p) v)

is smooth

Exercise . Let V C M^ be an n - dimensional linear subspace . The orthogonal projection of M^ onto V is the matrix n e M^x that satisfies

n = n2 = nT , imn = v . (2 . 6 . 1)

Prove that there is a unique matrix n e M^x satisfying Prove that , for every symmetric matrix S = ST e M^ , the kernel of S is the orthogonal complement of the image of S . If D e M^xn is any injective matrix whose image is V , prove that det (DTD) = 0 and

n = D (DtD) - 1Dt .

Theorem (Vector Bundles) . Let M C Mk be a smooth m - manifold and let E C M x M^ be a subset . Assume that , for every p e M , the set

Ep := jv e M^ | (p , v) e Ej

is an n - dimensional linear subspace of M^ . Let n : M ^ M^x be the map that assigns to each p e M the orthogonal projection of M^ onto Ep , i . e .

n (p) = n (p) 2 = n (p) T , imn (p) = Ep .

Then the following are equivalent .

E is a vector bundle .

For every p0 e M and every v0 e Ep0 there is a smooth map s : M ^ M^ such that s (po) = vo and s (p) e Ep for all p e M .

The map n : M ^ M^x is smooth .

For every p0 e M there is an open neighborhood U C M of p0 and a diffeomorphism n - 1 (U) ^ U x Mn : (p , v) ^ T (p , v) = (p , Tp (v)) such that the map Tp : Ep ^ Mn is an isometric isomorphism for all p e U .

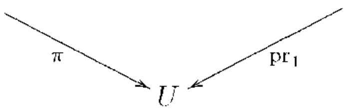
For every p0 e M there is an open neighborhood U C M of p0 and a diffeomorphism n - 1 (U) ^ U x Mn : (p , v) ^ T (p , v) = (p , Tp (v)

) such that the map Tp : Ep ^ Mn is a vector space isomorphism for all p e U .

Condition (i) implies that the projection n : E ^ M is a submersion . In (ii) the section s can be chosen to have compact support , i . e . there is a compact subset K C M such that s (p) = 0 for p / K .

Proof .

n - 1 (U) 5 » U x Rn .



Corollary . Let M c Rk be a smooth m - manifold . Then TM is a vector bundle over M and hence is a smooth 2m - manifold in Rk x Rk .

Proof . Let f : U ^ Q be a coordinate chart on an M - open set U c M with values in an open subset Q c Rm . Denote its inverse by ^ := f - 1 : Q ^ M . By Theorem the linear map d^ (x) : Rm ^ Rk is injective and its image is T^ (x) M for every x <= Q . Hence the map D : U ^ Rkxm defined by

D (p) := d^ (f (p)) <= Rkxm

is smooth and , for every p <= U , the linear map D (p) : Rm ^ Rk is injec - tive and its image is TpM . Thus the function nTM : M ^ Rkxk defined by with Ep = TpM is given by

nTM (p) = D (p) (D (p) TD (p)) 1 D (p) T for p <= U .

Hence nTM is smooth and so TM is a vector bundle by Theorem□

Let M c Rk be an m - manifold , N c R^ be an n - manifold , f : N ^ M be a smooth map , and E c M x Rd be a vector bundle . The pullback bundle is the vector bundle f *E ^ N defined by

f*E := { (q , v) <= N x Rd | v <= Ef{q)]

and the normal bundle of E is the vector bundle E± ^ M defined by

Ex := | (p , w) <= M x Rd l (v , w) = 0 V v <= Ep j .

Corollary . The pullback and normal bundles are vector bundles .

Proof . Let n = nE : M ^ Rdxd be the map defined by . This map is smooth by Theorem Moreover , the corresponding maps for f*E and EE are given by

nf*e = nE o f : n ^ Rdxd , nE± = n - nE : m ^ Rdxd .

These maps are smooth and hence it follows from Theorem that f*E and EE are vector bundles . □

Proof of Theorem . We first assume that E is a vector bundle and prove that n : E ^ M is a submersion . Let a : M ^ E denote the zero section given by a (p) := (p , 0) . Then n o a = id and hence it follows from the chain rule that the derivative dn (p , 0) : T (p , o) E ^ TpM is surjective . Now it follows that for every p <= M there is an e > 0 such that the derivative dn (p , v) : T (p , v) E ^ TpM is surjective for every v <= Ep with |v| < e . Consider the map f : E ^ E defined by

A (p , v) := (p , Av) .

This map is a diffeomorphism for every A > 0 . It satisfies

n = n o fx

and hence

dn (p , v) = dn (p , Av) o df\ (p , v) : T (p , v) E ^ TpM .

Since df\ (p , v) is bijective and dn (p , Av) is surjective for A < e / |v| it follows that dn (p , v) is surjective for every p <= M and every v <= Ep . Thus the projection n : E ^ M is a submersion for every vector bundle E over M .

We prove that (i) implies (ii) . Let po <= M and vo <= Ep0 . We have already proved that n is a submersion . Hence it follows from Lemma that there exists an M - open neighborhood U C M of p0 and a smooth map

a0 : U ^ E

such that

n o ao = id : U ^ U , ao (po) = (po , vo) .

Define the map so : U ^ R by

(p , so (p)) := ao (p) for p <= U .

Then so (po) = vo and so (p) <= Ep for all p <= U . Now choose e > 0 such that

{p <= M | |p - po| < e} C U

and choose a smooth cutoff function ft : Rk ^ [0 , 1] such that ft (po) = 1 and ft (p) = 0 for |p — po| > e . Define s : M ^ R by

s (w) = / ^oOp^ if p <= U (p) :\ 0 , if p <= U .

This map satisfies the requirements of (ii) .

We prove that (ii) implies (iii) . Thus we assume that E satisfies (ii) . Choose p0 G M and a basis v\ , . . . , vn of Ep0 . By (ii) there exists smooth sections s\ , . . . , sn : M ^ R of E such that Si (p0) = Vi for i = 1 , . . . , n . Now there exists an M - open neighborhood U C M of p0 such that the vec - tors si (p) , . . . , sn (p) are linearly independent , and hence form a basis of Ep for every p G U . Hence , for every p G U , we have

Ep = imD (p) , D (p) := [si (p) ■ ■ ■ sn (p)] G R^xn .

By Exercise this implies n (p) = D (p) (D (p) TD (p)) - 1D (p) T for every p G U .

Thus every p0 G M has a neighborhood U such that the re - striction of n to U is smooth . This shows that (ii) implies (iii) .

We prove that (iii) implies (iv) . Fix a point p0 G M and choose a basis

v1 , . . . , vn of Ep0 . For p G M define

D (p) := [n (p) vi ■ ■ ■ n (p) vn] G R^xn

Then D : M ^ R^xn is a smooth map and D (p0) has rank n . Hence the set

U := {p G M | rankD (p) = n} C M

is an open neighborhood of p0 and Ep = imD (p) for all p G U . Thus

n - 1 (U) = { (p , v) G E | p G U} C E

is an open set containing n - 1 (p0) . Define the map T : n - 1 (U) ^ U x Rn by

T (p , v) := (p , Tp (v)) , Tp (v) := (D (p) TD (p)) 7D (p) Tv for p G U and v G Ep . This map is bijective and its inverse is given by

T - 1 (p , <=) = (p , T - 1 (<=)) , T - 1 (<=) = D (p) (D (p) TD (p)) - 1 / 2<=

for p G U and <= G Rn . Thus T is a diffeomorphism and |Tp (v) | = |v| for all p G U and all v G Ep . This shows that (iii) implies (iv) .

That (iv) implies (v) is obvious .

We prove that (v) implies (i) . Shrinking U if necessary , we may as - sume that there exists a coordinate chart 0 : U ^ Q with values in an open set Q C Rm . Then the composition (0 x id) o T : n - 1 (U) ^ Q x Rn is a dif - feomorphism . Thus E C Rk x R^ is a manifold of dimension m + n and this proves □

Exercise . Construct a vector bundle E C S1 x R2 of rank 1 that does not admit a global trivialization , i . e . that is not isomorphic to the trivial bundle S1 x R . Such a vector bundle is called a Mobius strip . Define the notion of an isomorphism between two vector bundles E and F over M .

4.3 THE IMPLICIT FUNCTION THEOREM

In this subsection we carry over the Implicit Function Theorem in

Corollary to smooth maps on vector bundles .

Theorem (Implicit Function Theorem) .

Let M c Rk be a smooth m - manifold , let N c Rk be a smooth n - manifold , let E c M x R be a smooth vector bundle of rank n , let W c E be open , and let f : W ^ N be a smooth map . For p e M define fp : Wp ^ N by

Wp := {v e Ep | (p , v) e W} , fp (v) := f (p , v) .

Let po e M such that 0 e Wp0 and dfp0 (0) : Tp0 M ^ Tq0N is bijective , where q0 := f (p0 , 0) e N . Then there exists a constant e > 0 , open neighbor - hoods U0 c M of p0 and V0 c N of q0 , and a smooth map h : U0 x V0 ^ such that { (p , v) e E | p e U0 , |v| < e} c W and

h (p , q) e Ep , |h (p , q) | < e

for all (p , q) e U0 x V0 and

fp (v) = 0   v = h (p , q)

for all (p , q) e U0 x V0 , and all v e Ep with |v| < e .

Proof . Choose a coordinate chart ^ : V ^ Rn on an open set V c N con - taining q0 . Choose an open neighborhood U c M of p0 such that (p , 0) e W and f (p , 0) e V for all p e U , there is a coordinate chart 0 : U ^ Q c Rm , and there is a local trivialization T : n - 1 (U) ^ U x Rn as in Theorem 2 . 6 . 8 with |Tp (v) | = |v| for p e U and v e Ep . Define Br := {<= e Rn | |<=| < r} and choose r > 0 so small that T - 1 (U x Br) c W and f o T - 1 (U x Br) c V . Define the map F : Q x Rn x Br ^ Rn by

F (x , y , <=) := ^ o f o T - 1 (0 - 1 (x) , <=) - y

for (x , y) e Q x Rn and <= e Br . Let x0 := 0 (p0) and y0 := ^ (q0) . Then we have F (x0 , y0 , 0) = 0 and the derivative d3F (x0 , y0 , 0) : Rn ^ Rn of F with respect to <= at (x0 , y0 , 0) is bijective . Hence Corollary A . 2 . 6 asserts that there exist open neighborhoods U0 c U of p0 and V0 c V of q0 , a con - stant 0 < e < r , and a smooth map g : 0 (U0) x ■0 (V0) ^ Be such that

F (x , y , <=) = 0   g (x , y) = <=

for all (x , y) e 0 (U0) x ^ (V0) and all <= e Be . Thus the map

h : U0 x V0 ^ R* , h (p , q) := T - 1 (g (0 (p) , ^ (q))) ,

satisfies the requirements of Theorem □

The Theorem of Frobenius

Let M c Rk be an m - dimensional manifold and n be a nonnegative integer . A subbundle of rank n of the tangent bundle TM is a subset E c TM that is itself a vector bundle of rank n over M , i . e . it is a submanifold of TM and the fiber Ep = {v e TpM | (p , v) e E} is an n - dimensional linear subspace of TpM for every p e M . Note that the rank n of a subbundle is necessarily less than or equal to m . In the literature a subbundle of the tangent bundle is sometimes called a distribution on M . We shall , however , not use this terminology in order to avoid confusion with the concept of a distribution in the functional analytic setting .

Definition . Let M c Rk be an m - dimensional manifold and E c TM be a subbundle of rank n . The subbundle E is called involutive if , for any two vector fields X , Y e Vect (M) , we have

X (p) , Y (p) e Ep Vp e M =^ [X , Y] (p) e Ep Vp e M .

The subundle E is called integrable if , for every p0 e M , there exists a submanifold N c M such that p0 e N and TpN = Ep for every p e N . A foliation box for E is a coordinate chart $: U ^ Q on an M - open subset

U c M with values in an open set Q c Rn x Rm - n such that the set Q n (Rn x {y}) is connected for every y e Rm - n and , for every p e U and every v e TpM , we have

v e Ep d$ (p) v e Rn x {0} .

Theorem (Frobenius) . Let M c Rk be an m - dimensional manifold , and E c TM be a subbundle of rank n . Then the following are equivalent .

E is involutive .

E is integrable .

For every p0 e M there is a foliation box $: U ^ Q with p0 e U . It is easy to show that (iii) =^ (ii) =^ (i) (see below) . The hard part of the theorem is to prove that (i) (iii) . This requires the following lemma .

Lemma . Let E c TM be an involutive subbundle and X e Vect (M) be a complete vector field such that X (p) e Ep for every p e M . Denote by

R ^ Diff (M) : t ^ ft

the flow of X . Then , for all t e R and p e M , we have

dft (p) Ep = E^ .

Lemma Let po e M , choose a foliation box f : U ^ Q for E with p0 e U , and define

N := (p e U | f (p) e Rn x {yo}}

where (x0 , y0) := f (p0) e Q . Then N satisfies the requirements of (ii) .

We prove that (ii) implies (i) . Choose two vector fields X , Y e Vect (M) that satisfy X (p) , Y (p) e Ep for all p e M and fix a point p0 e M . Then , by (ii) , there exists a submanifold N c M containing p0 such that TpN = Ep for every p e N . Hence the restrictions X|N and Y|N are vector fields on N and so is the restriction of the Lie bracket [X , Y] to N . Thus we have [X , Y] (p0) e Tp0N = Ep0 as claimed .

We prove that (i) implies (iii) . Thus we assume that E is an involutive subbundle of TM and fix a point p0 e M . By Theorem there exist vector fields Xi , . . . , Xn e Vect (M) such that Xj (p) e Ep for all i and p and the vectors X1 (p0) , . . . , Xn (p0) form a basis of Ep0 . Using Theorem again we find vector fields Y1 , . . . , Ym - n e Vect (M) such that the vectors

Xi (p0) , . . . , Xn (p0) , Yi (p0) , . . . , Ym - n (p0)

form a basis of Tp0M . Using cutoff functions as in the proof of Theorem we may assume without loss of generality that the vector fields Xj and Yj have compact support and hence are complete . Denote by fi , . . . , $n the flows of the vector fields X1 , . . . , Xn , respectively , and by 44 , . . . , - n the flows of the vector fields Y1 , . . . , Ym - n . Define the map

^ : Rn x Rm - n ^ M

by

By Lemma , this map satisfies

dE

dx (x , y) G Ep (x , y)

for all x G Rn and y G Rm - n . Moreover ,

g (o . °) = Xi{pa) . (o . °) = r , (Kl) ,

and so the derivative

(0 , 0) : Rn x Rm - n ^ Tp0M

is bijective . Hence , by the Inverse Function Theorem there is an open neighborhood Q c Rn x Rm - n of the origin such that the set

U := E (Q) C M

is an M - open neighborhood of p0 and E|n : Q ^ U is a diffeomorphism . Thus the vectors d^ / dx^ (x , y) are linearly independent for every (x , y) G Q and , by form a basis of E^ (x>y) . Hence

f := (E|n) - 1 : U ^ Q is a foliation box and this proves Theorem□

To complete the proof of the Frobenius theorem it remains to prove Lemma This requires the following result .

Lemma . Let E C TM be an involutive subbundle . If ft : R2 ^ M is a smooth map such that

dB dB

— (s 0) G Ep (s , o) , - djf (s , t) G Eh (s>t) ,

for all s , t G R then

3B

— (s , t) G Ep (s , t) ,

for all s , t G R .

Lemma Let X e Vect (M) be a complete vector field satisfying X (p) e Ep for every p e M and let ftt be the flow of X . Choose a point po e M

Notes

and a vector v0 e Ep0 . By Theorem there is a vector field Y e Vect (M) with values in E such that Y (p0) = v0 . Moreover this vector field may be chosen to have compact support and hence it is complete Thus there is a solution 7 : R ^ M of the initial value problem

Y (s) = Y (Y (s)) Y (0) = Po .

Define - : R2 ^ M by

- (s , t) := ft (Y (s))

for s , t e R . Then

Is (s , 0) = 7 (s) = Y (Y (s)) e Efj (s , 0) ,

d - (s , t) = X (P (s , t)) e Ef (s , t) for all s , t e R . Hence it follows from Lemma that

d<=f

ds

for every t e R . □

Proof of Lemma . Given any point p0 e M we choose a coordinate chart ft : U ^ H , defined on an M - open set U C M with values in an open set H C Rn x Rm - n , such that p0 e U and dft (p0) EP0 = Rn x {0} . Shrinking U , if necessary , we obtain that dft (p) EP is the graph of a matrix A e R (m - n) xn for every p e U . Thus there is a map A : H ^ R (m - n) xn such that , for every p e U , we have

dft (p) Ep = { (e , A (x , y) C) | e e Rn} , (x , y) := ft (p) e H . (2 . 7 . 6)

For (x , y) e H we define the linear maps 0 , 4 OA

d - (x , y) : Rn ^ R (m - n) xn , (x , y) : Rm - n ^ R (m - n) xn

dx dy

by

( ) e ^ edA ( ) dA ( ) m - 1 dA ( )

dx (x , y) ■e := <= e dxi (x , y) , w (x , y) ■n := g dj (x , y) ,

for e = (Ci , . . . , en) e Rn and y = (y^ . . . , ym - n) e Rm - n .

Claim 1 . Let (x , y) G Q , Z , Z' G Rn and define n , n' G Rm n by n := A (x , y) Z and n' := A (x , y) Z' . Then

(dA , , . dA \ \ (dA , dA \

\dx y ■ Z + w (x'y) ■ "T = lte (x'y) ■ Z + w (x'y) ■") Z

The graphs of the matrices A (z) determine a subbundle E c Q x Rm with fibers

Ez := { (Z , n) G Rn X Rm - n | n = A (x , y) Z} for z = (x , y) G Q . This subbundle is the image of

E|u := { (p , v) | p G U , v G Ep}

under the diffeomorphism TM|u ^ Q x Rm : (p , v) ^ (fi (p) , dfi (p) v) and hence it is involutive . Now define the vector fields Z , Z' : Q ^ Rm by

Z (z) := (Z , A (z) Z) , Z' (z) := (Z' , A (z) Z') , z G Q .

Then Z and Z' are sections of E and their Lie bracket [Z , Z'] is given by

[Z , Z'] (z) = l0 , (dA (z) Z' (z)) Z (z) - (dA (z) Z (z)) C' (z)) .

Since E is involutive the Lie bracket [Z , Z'] must take values in the graph of A .

Claim 2 . Let I , J c R be open intervals and z = (x , y) : 12 ^ Q be a smooth map . Fix two points so G I and to G J and assume that

dS (so , to) = A (x (so , to) , y (so , to)) dS (so , to) ,

ddt (s , t) = A (x (s , t) , y (s , t)) | (s , t)

for all s G I and t G J . Then

dS (so , t) = A (x (so , t) , y (so , t)) dS (so , t)

for all t G J .

Notes

Equation holds by assumption for t = to . Moreover , dropping the argument z (so , t) = z = (x , y) for notational convenience we obtain

d f dy . dx \ d2y . d2x f dA dx dA dy \ dx

dt \ds ds J dsdt dsdt \ dx dt + dy dt J ds

d2y ^ d2x f dA dx dA f ^ dx ^ dx

dsdt dsdt \dx dt dy \ dt)) ds d2y ^ d2x f dA dx dA f ^ dx ^ dx

dsdt dsdt \dx ds dy \ ds)) dt d2y ^ d2x f dA dx ^dA dy \ dx

dsdt dsdt \ dx ds dy ds J dt

fdA fdy . dx\\ dx + \ dy \ ds ds j j dt

dA f dy dx \\ dx

dy \ ds ds J J dt

Here the second step follows from the third equation follows from Claim 1 , and the last step follows by differentiating equation with respect to s . Define p : J ^ Rm - n by

^ (t) := Ids (so , t) - A (x (so , t) , y (so , t)) (so , t) .

By and what we have just proved , the function p satisfies the linear differential equation

n (t) = (dy (x (so , t) , y (^t)) ■ n (t)) ^ (so , t) , n (to) = 0 .

Hence p (t) = 0 for all t e J . This proves and Claim 2 .

Now let P : R2 ^ M be a smooth map satisfying and fix a real number so . Consider the set W := {t e R dsP (so , t) e Ep (s0>t) } . By going to local coordinates , we obtain from Claim 2 that W is open . Moreover , W is obviously closed , and W = 0 because 0 e W by Hence W = R . Since so e R was chosen arbitrarily . □

Any subbundle E c TM determines an equivalence relation on M via

there is a smooth curve 7 : [0 , 1] ^ M (27i0)

po Pi => such that 7 (0) = po , 7 (1) = p\ , Y (t) e EY (t) V t .)

If E is integrable this equivalence relation is called a foliation and the equivalence class of po e M is called the leaf of the foliation through po

Example . Consider the torus M := S1 x S1 C C2 with the tangent bundle

TM = { (zi , z2 , iAiZi , iA2^2) <= C4 | |zi| = |z2| = 1 , Ai , A2 <= R} . Let wi , w2 be real numbers and consider the subbundle

E := { (zi , Z2 , itwizi , UW2Z2) <= C4 | |zi| = |Z21 = 1 , t <= R} .

The leaf of this subbundle through z = (zi , z2) <= T2 is given by

It is a submanifold if and only if the quotient wi / u2 is a rational number (or w2 = 0) . Otherwise each leaf is a dense subset of T2 .

Exercise . Consider the manifold M = S3 C R4 = C2 and define

E :={ (z , Z) <= C2 x C2 | |z| = 1 , Z A z , iZ T z} .

Thus the fiber

Ez C Tz S3 = z±

is the maximal complex linear subspace of TzS3 . Prove that E has real rank 2 and is not integrable .

Exercise . Let E C TM be an involutive subbundle of rank n and let L C M be a leaf of the foliation determined by E . A subset V C L is called L - open if it can be written as a union of sub manifolds N of M with tangent spaces TpN = Ep for p <= N . Prove that the L - open sets form a topology on L (called the intrinsic topology) . Prove that the obvious inclusion 1 : L ^ M is continuous with respect to the intrinsic topology on L . Prove that the inclusion 1 : L ^ M is proper if and only if the intrinsic topology on L agrees with the relative topology inherited from M (called the extrinsic topology) .

The Intrinsic Definition of a Manifold*

It is somewhat restrictive to only consider manifolds that are embedded in some Euclidean space . Although we shall see that (at least) every compact manifold admits an embedding into a Euclidean space , such an embedding is in many cases not a natural part of the STRUCTURE of a manifold . In particular , we encounter manifolds that are described as quotient spaces and there are manifolds that are embedded in certain infinite dimensional Hilbert spaces . For this reason it is convenient , at this point , to introduce a more general intrinisc definition of a manifold . 5 This requires some background from point set topology that is not covered in the first year analysis courses . We shall then see that all the definitions and results of this chapter carry over in a natural manner to the intrinsic setting . We begin by recalling the intrinsing definition of a smooth manifold in .

Definition (Smooth m - Manifold) . Let m <= No and M be a set . A chart on M is a pair ($, U) where U C M and $ is a bijection from U to an open set $ (U) C Rm . Two charts ($1 , U1) , ($2 , U2) are called com - patible iff $1 (U1 fl U2) and $2 (U1 n U2) are open and the transition map

$21 = $2 ◦ 1: $1 (U1 n U2) ^ $2 (U1 n U2)

is a diffeomorphism . A smooth atlas on M is a collection A of charts on M any two of which are compatible and such that the sets U , as ($, U) ranges over A , cover M (i . e . for every p <= M there is a chart ($, U) <= A with p <= U) . A maximal smooth atlas is an atlas which contains every chart which is compatible with each of its members . A smooth m - manifold is a pair consisting of a set M and a maximal atlas A on M .

In Lemma it was shown that , if A is an atlas , then so is the collection A of all charts compatible with each member of A . Moreover , the atlas A is maximal , so every atlas extends uniquely to a maximal atlas . For this reason , a manifold is usually specified by giving its underlying set M and some atlas on M . Generally , the notation for the atlas is suppressed and the manifold is denoted simply by M . The members of the atlas are called coordinate charts or simply charts on M . By Lemma a smooth m - manifold admits a unique topology such that , for each chart (0 , U) of the smooth atlas , the set

U C M

---

is open and the bijection

0 : U ^ 0 (U)

is a homeomorphism onto the open set 0 (U) C Rm . This topology is called the intrinsic topology of M and is described in the following definition .

Definition Let M be a smooth m - manifold . The intrinsic topology

on the set M is the topology induced by the charts , i . e . a subset

W C M

is open in the intrinsic topology iff 0 (U n W) is an open subset of Rm for every chart (0 , U) on M .

Example . The complex projective space CPn is the set

CPn = {<= C Cn + 111 is a 1 - dimensional complex subspace} of complex lines in Cn + 1 . It can be identified with the quotient space

CPn = (Cn + 1 \ {0}) / C*

of nonzero vectors in Cn + 1 modulo the action of the multiplicative group C* = C \ {0} of nonzero complex numbers . The equivalence class of a nonzero vector z = (z0 , . . . , zn) <= Cn + 1 will be denoted by

[z] = [zo : Z1 : ■ ■ ■ : Zn] := {Az | A <= C*}

and the associated line is I = Cz . An atlas on CPn is given by the open cover Ui := {[z0 : ■ ■ ■ : zn] | zi = 0} for i = 0 , 1 , . . . , n and the coordinate charts 0i : Ui ^ Cn are

«[zo : ■ ■ ■ : zn]) := (z0 , . . . , ^ , zi±1 , . . . , .

zi zi zi zi

Exercise: Prove that each 0j is a homeomorphism and the transition maps are holomorphic . Prove that the manifold topology is the quotient topology , i . e . if n : Cn + 1 \ {0} ^ CPn denotes the obvious projection , then a sub - set U C CPn is open if and only if n - 1 (U) is an open subset of Cn + 1 \ {0} .

Example . The real projective space RPn is the set

RPn = {! C Rn + 111 is a 1 - dimensional linear subspace}

of real lines in Rn + 1 . It can again be identified with the quotient space

RPn = (Rn + 1 \ {0}) / R*

of nonzero vectors in Rn + 1 modulo the action of the multiplicative group R* = R \ {0} of nonzero real numbers , and the equivalence class of a nonzero vector x = (x0 , . . . , xn) <= Rn + 1 will be denoted by

[x] = [x0 : x1 : ■ ■ ■ : xn] := {Ax | A <= R*} .

An atlas on RPn is given by the open cover

Uj := {[xo : ■ ■ ■ : xn] | xj = 0}

and the coordinate charts 0i : Ui ^ Rn are again given by , with zj replaced by xj .

Example . The real n - torus is the topological space

Tn := Rn / Zn

equipped with the quotient topology . Thus two vectors x , y e Rn are equivalent if their difference x — y e Zn is an integer vector and we denote by n : Rn ^ Tn the obvious projection which assigns to each vector x e Rn its equivalence class

n (x) := [x] := x + Zn .

Then a set U C Tn is open if and only if the set n - 1 (U) is an open subset of Rn . An atlas on Tn is given by the open cover

Ua := {[x] | x e Rn , |x — a\ < 1 / 2} ,

parametrized by vectors a e Rn , and the coordinate charts <fia : Ua ^ Rn defined by 0a ([x]) := x for x e Rn with \x — a\ < 1 / 2 . Exercise: Show that each transition map for this atlas is a translation by an integer vector .

Example . Consider the complex Grassmannian

Gk (Cn) := {V C Cn \ v is a k - dimensional complex linear subspace} .

This set can again be described as a quotient space Gk (Cn) = Fk (Cn) / U (k) . Here

Fk (Cn) := {D e Cnxk \ D*D = ll}

denotes the set of unitary k - frames in Cn and the group U (k) acts on Fk (Cn) contravariantly by D ^ Dg for g e U (k) . The projection

n : Fk (Cn) ^ Gk (Cn)

sends a matrix D e Fk (Cn) to its image V := n (D) := im D . A subset U C Gk (Cn) is open if and only if n - 1 (U) is an open subset of Fk (Cn) . Given a k - dimensional subspace V C Cn we can define an open set UV C Gk (Cn) as the set of all k - dimensional subspaces of Cn that can be represented as graphs of linear maps from V to V^ . This set of graphs can be identified with the complex vector space HomC (V , V^) of complex linear maps from V to V^ and hence with C (n - k) xk . This leads to an atlas on Gk (Cn) with holomorphic transition maps and shows that Gk (Cn) is a manifold of complex dimension kn — k2 . Exercise: Verify the details of this construction . Find explicit formulas for the coordinate charts and their transition maps . Carry this over to the real setting . Show that CPn and RPn are special cases .

Example (The real line with two zeros) . A topological space M is called Hausdorff if any two points in M can be separated by disjoint open neighborhoods . This example shows that a manifold need not be a Hausdorff space . Consider the quotient space

M := R x {0 , 1} / =

where [x , 0] = [x , 1] for x = 0 . An atlas on M consists of two coordinate charts 0o : U0 ^ R and : Ui ^ R where

Ui := {[x , i] | x <= R} , 0j ([x , i]) := x

for i = 0 , 1 . Thus M is a 1 - manifold . But the topology on M is not Hausdorff , because the points [0 , 0] and [0 , 1] cannot be separated by disjoint open neighborhoods .

Notes

Example . Consider the vector space X = R x R2 with the equivalence relation

[ti . xi . wi=[t2 . x2 . y2] ~ otf yiy = y20= . 0 - % - + :iyi=t2 + xm

or yi = y2 = 0 , 11 = t2 , xi = x2 .

For y = 0 we have [0 , x , y] = [t , x — t / y , y] , however , each point (x , 0) on the x - axis gets replaced by the uncountable set R x{ (x , 0) } . Our manifold is the quotient space M := X / = . This time we do not use the quotient topology but the topology induced by our atlas Definition . The coordinate charts are parametrized by the reals: for t <= R the set Ut C M and the coordinate chart 0t : Ut ^ R2 are given by

Ut := {[t , x , y] | x , y <= R} , & ([t , x , y]) := (x , y) .

A subset U C M is open , by definition , if 0t (U n Ut) is an open subset of R2 for every t <= R . With this topology each 0t is a homeomorphism from Ut onto R2 and M admits a countable dense subset S := {[0 , x , y] | x , y <= Q} . However , there is no atlas on M consisting of countably many charts . (Each coordinate chart can contain at most countably many of the points [t , 0 , 0] .) The function f : M ^ R given by f ([t , x , y]) := t + xy is smooth and each point [t , 0 , 0] is a critical point of f with value t . Thus f has no regular value . Exercise: Show that M is a path - connected Hausdorff space .

In Theorem we will show that smooth manifolds whose topology is Hausdorff and second countable are precisely those that can be embedded in Euclidean space . Most authors tacitly assume that manifolds are Hausdorff and second countable and so will we after the end of the present chapter . However before there is no need to impose these hypotheses .

Smooth Maps and Diffeomorphisms

Our next goal is to carry over all the definitions from embedded manifolds in Euclidean space to the intrinsic setting .

Definition (M , { (0_a , U_a) $\}_{a(E}$A) , (N , { (tp , Vp) W) be smooth manifolds . A map f : M ^ N is called smooth if it is continuous and the map is smooth for every a <= A and every ft <= B . It is called a diffeomorphism if it is bijective and f and f^{-1} are smooth . The manifolds M and N are called diffeomorphic if there exists a diffeomorphism f : M ^ N .

The reader may check that the notion of a smooth map is independent of the atlas used in the definition , that compositions of smooth maps are smooth , and that sums and products of smooth maps from M to $\mathbb{R}$ are smooth .

Check your Progress - 1

Discuss Vector Bundles

Discuss Submersions

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4.4LET US SUM UP

In this unit we have discussed the definition and example of Vector Bundles And Submersions , The Implicit Function Theorem

4.5 KEYWORDS

Vector Bundles And Submersions …. Submersions Let M c Rk be a smooth m - manifold and N C R be a smooth n - manifold . A smooth map f : N ^ M is called a submersion if its derivative

The Implicit Function Theorem …..In this subsection we carry over the Implicit Function Theorem in Corollary to smooth maps on vector bundles .

4.6 QUESTIONS FOR REVIEW

Explain Vector Bundles

Explain Submersions

4.7 ANSWERS TO CHECK YOUR PROGRESS

Vector Bundles (answer for Check your Progress - 1 Q)

Submersions (answer for Check your Progress - 1 Q)

4.8 REFERENCE

Differential Geometry, Differential Geometry & Application, Introduction to Defferential Geometry, Basic of Differential Geometry.

UNIT-V : GEODESICS

STRUCTURE

5.0 Objectives

5.1 Introduction

5.2 Geodesics

5.3 Exponential Map

5.4 Let Us Sum Up

5.5 Keywords

5.6 Questions For Review

5.7 Answers To Check Your Progress

5.8 References

5.0 OBJECTIVES

After studying this unit , you should be able to:

- Understand about Geodesics
- Exponential Map

5.1 INTRODUCTION

Differential geometry arose and developed as a result of and in connection to the mathematical analysis of curves and surfaces Mathematical analysis of curves and surfaces had been developed to answer some of unanswered questions that appeared in calculus like the reasons for relationships between complex shapes and curves , series and analytic functions Geodesics , Exponential Map

5.2 GEODESICS

This chapter introduces Geodesics in Riemannian manifolds . It begins in by introducing geodesics as extremals of the energy and length functionals and characterizing them as solutions of a second order differential equation . In we show that minimizing the length with fixed endpoints gives rise to an intrinsic distance function d : M x M ^ r which induces the topology M inherits from the ambient space rn . introduces the exponential map , shows that geodesics minimize the length on short time intervals , establishes the existence of geodesically convex neighborhoods , and shows that the geodesic flow is complete if and only if (M , d) is a complete metric space , and that in the complete case any two points are joined by a minimal geodesic . discusses geodesics in the intrinsic setting .

Length and Energy

The concept of a geodesic in a manifold generalizes that of a straight line in Euclidean space . A straight line has parametrizations of form t ^ p + a (t) v where a : r ^ r is a diffeomorphism and p , v <= rn . Different choices of a yield different parametrizations of the same line . Certain parametrizations are preferred , for example those parametrizations which are "proportional to the arclength" , i . e . where a (t) = at + b for constants a , b <= r , so that the tangent vector a (t) v has constant length . The same distinctions can be made for geodesics . Some authors use the term geodesic to include all parametrizations of a geodesic while others restrict the term to cover only geodesics parametrized proportional to arclength . We follow the latter course , referring to the more general concept as a "reparametrized geodesic" . Thus a reparametrized geodesic need not be a geodesic .

We assume throughout that M C rn is a smooth m - manifold .

Definition (Length and Energy) . Let I =[a , b] C r be a com - pact interval with a < b and let 7 : I ^ M be a smooth curve in M . The length L (y) and the energy E (7) are defined by

L (7) := f |'y (t) |

J a

1 r b

E (y) := 2 Ja l^ (t) |2 dt .

A variation of 7 is a family of smooth curves 7s : I ^ M , where s ranges over the reals , such that the map r x I ^ M : (s , t) ^ Ys (t) is smooth and

Yo = 7 .

The variation (7s}seR is said to have fixed endpoints if 7s (a) = 7 (a) and Ys (b) = 7 (b) for all s e r .

Remark . The length of a continuous function 7 : [a , b] ^ rn can be defined as the supremum of the numbers ^N1 |y (tj) — 7 (ti - i) | over all partitions a = t0 <t1 < ••• <tN = b of the interval [a , b] . By a theorem in first year analysis this supremum is finite whenever 7 is continuously differentiable .

We shall sometimes suppress the notation for the endpoints of a , b e I . When y (a) = p and 7 (b) = q we say that 7 is a curve from p to q . One can always compose 7 with an affine reparametrization t' = a + (b — a) t to obtain a new curve Y' (t) := Y (t') on the unit interval 0 < t < 1 . This new curve satisfies L (y') = L (y) and E (y') = (b — a) E (7) . More generally , the length L (y) , but not the energy E (y) , is invariant under reparametrization .

Remark (Reparametrization) . Let I = [a , b] and I' = [a' , b'] be compact intevrals . If 7 : I ^ rn is a smooth curve and a : I' ^ I is a smooth function such that a (a') = a , a (b') = b , and a (t) > 0 for all t e I' , then

L (Y o a) = l (y) .

To see this , we compute

Here second equation follows from the chain rule and the fact that a (t') > 0 for all t' e [a' , b'] , and the third equation follows from the change of variables formula for the Riemann integral .

Notes

Theorem (Characterization of Geodesics) . Let I =[a , b] C r be a compact interval and let y : I m M be a smooth curve . Then the following are equivalent .

(i) y is an extremal of the energy functional , i . e .

d

e (Y s) = 0

s=0

for every variation (Ys}seR of y with fixed endpoints .

(ii) y is parametrized proportional to the arclength , i . e . the veloc - ity |Y (t) | = c > 0 is constant , and either y is constant , i . e . y (t) = p = q for all t <= I , or c > 0 and y is an extremal of the length functional , i . e .

d

ds

L (Y s) = 0

s=0

for every variation (Ys}seR of y with fixed endpoints .

The velocity vector of y is parallel , i . e . VY (t) = 0 for all t <= I .

The acceleration of y is normal to M , i . e . Y (t) L TY (t) M for all t <= I .

If ($, y , y') is a development of M along M' = rm , then y' : I m Rm is a straight line parametrized proportional to the arclength , i . e . 7' = 0 .

Definition (Geodesic) . A smooth curve y : I m M on an interval I is called a geodesic if its restriction to each compact subinterval satisfies the equivalent conditions of Theorem So y is a geodesic if and only if

VY (t) = 0 for all t <= I .

By the Gaufi - Weingarten formula with X = y this is equivalent to

Y (t) = h7 (t) (7 (t) , Y (t)) for all t <= I .

Remark (i) The conditions (i) and (ii) in Theorem are meaningless when I is not compact because then the curve has at most one endpoint and the length and energy integrals may be infinite . However , the conditions (iii) , (iv) , and (v) in Theorem are equivalent for smooth curves on any interval , compact or not .

(ii) The function s m E (ys) associated to a smooth variation is always smooth and so condition (i) in Theorem is meaningful . However , more care has to be taken in part (ii) because the function s m L (ys) need not be differentiable . However , it is differentiable at s = 0 whenever y (t) = 0 for all t I .

ds

The Space of Paths

Fix two points p , q <= M and a compact interval I = [a , b] and denote by

Qp , q := Qp , q (I) := {7 : I ^ M | 7 is smooth and 7 (a) = p , 7 (b) = q}

the space of smooth curves in M from p to q , defined on the interval I . Then the length and energy are functional L , E : Qp , q ^ r and their extremal points can be understood as critical points as we now explain .

We may think of the space Qp , q as a kind of "infinite dimensional man - ifold" . This is to be understood in a heuristic sense and we use these terms here to emphasize an analogy . Of course , the space Qp , q is not a manifold in the strict sense of the word . To begin with it is not embedded in any finite dimensional Euclidean space . However , it has many features in common with manifolds . The first is that we can speak of smooth curves in Qp , q . Of course Qp , q is itself a space of curves in M . Thus a smooth curve in Qp , q would then be a curve of curves , namly a map r ^ Qp , q : s ^ ys that as - signs to each real number s a smooth curve Ys : I ^ M satisfying 7s (a) = p and ys (b) = q . We shall call such a curve of curves smooth if the associated map r x I ^ M : (s , t) ^ 7s (t) is smooth . Thus smooth curves in Qp , q are the variations of 7 with fixed endpoints introduced

Notes

Having defined what we mean by a smooth curve in Qp , q we can also differentiate such a curve with respect to s . Here we can simply recall that , since M c rn , we have a smooth map r x I ^ rn and the derivative of the curve s ^ Ys in Qp , q can simply be understood as the partial derivative of the map (s , t) ^ 7s (t) with respect to s . Thus , in analogy with embed - ded manifolds , we define the tangent space of the space of curves Qp , q at y as the set of all derivatives of smooth curves r ^ Qp , q : s ^ Ys passing through y , i . e .

Ys r ^ Qp , q : s ^ ys is smooth and y0 = 7

Let us denote such a partial derivative by X (t) := JS| 0 Ys (t) <= T7 (- t) M . Thus we obtain a smooth vector field along 7 . Since Ys (a) = p and 7S (b) = q for all s , this vector field must vanish at t = a , b . This suggests the formula

T7Qp , q = {X e Vect (Y) | X (a) = 0 , X (b) = 0} .

That every tangent vector of the path space Qp , q at 7 is a vector field along 7 vanishing at the endpoints follows from the above discussion . The converse inclusion is the content of the next lemma .

Lemma . Let p , q <= M , y <= Qp , q , and X <= Vect (Y) with X (a) = 0 and X (b) = 0 . Then there exists a smooth map r ^ Qp>q : s ^ 7s such that

Yo (t) = Y (t) qS

Ys (t) = X (t) for all t <= I .

s=0

Proof . The proof has two steps .

Step 1 . There exists smooth map M x I ^ rn : (r , t) ^ Yt (r) with compact support such that Yt (r) <= TrM for all t <= I and r <= M , Yt (r) = 0 for all t <= I and all r <= M \ K , and Ya (r) = Yb (r) = 0 for all r <= M .

Define Zt (r) := n (r) X (t) for t <= I and r <= M . Choose an open set U C rn such that y (I) C U and U n M is compact (e . g . take U := (

Ja<t<b B<= (Y (t)) for e > 0 sufficiently small) . Now let ft : rn ^ [0 , 1] be a smooth cutoff function with support in the unit ball such that , 0 (0) = 1 and define the vector fields Yt by Yt (r) := ft (e - 1 (r — y (t))) Zt (r) for t <= I and r <= M .

Step 2 . We prove the lemma .

The vector field Yt : M ^ TM in Step 1 is complete for each t . Thus there exists a unique smooth map r x I ^ M : (s , t) ^ Ys (t) such that , for each t <= I , the curve r ^ M : s ^ Ys (t) is the unique solution of the differential equation ■§ - Ys (t) = Yt (Ys (t)) with Y0 (t) = Y (t) . These maps Ys satisfies by Step 1 . □

We can now define the derivative of the energy functional E at y

in the direction of a tangent vector X <= TYQp , q by

m . Y) X := d

E (Ys) , (4 . 1 . 8)

s=0

where s ^ Ys is as in Lemma Similarly , the derivative of the length functional L at y in the direction of X <= TYQp , q is defined by

L (Ys) .

s=0

To define the functions s ^ E (ys) and s ^ L (ys) must be differentiable at s = 0 . This is true for E but it only holds for L when 7 (t) = 0 for all t <= I . Second , we must show that the right hand sides of depend only on X and not on the choice of {ys}seR . Third , we must verify that dE (y) : TYQp>q ^ r and dL (Y) : Dp>q ^ r are linear maps . This is an exercise in first year analysis . A curve y <= ^p , q is is then an extremal point of E (respectively L when 7 (t) = 0 for all t) if and only if dE (y) = 0 (respectively dL (Y) = 0) . Such a curve is also called a critical point of E (respectively L) .

Characterization of Geodesics

Notes

Proof of Theorem The equivalence of (iii) and (iv) follows directly from the equations Vft (t) = n (7 (t)) ft (t) and ker (n (y (t))) = TY (t) Mx .

We prove that (i) is equivalent to (iii) and (iv) . Let X e TYQp , q and choose a smooth curve of curves r ^ Qp , q : s ^ satisfying . Then the function (s , t) ^ |fts (t) |2 is smooth and hence

dE (Y) x=dS

1 fb

l7s (t) | dt |'is (t) |2 dt

s

s=0

/ (

[j (t) , X (t)) dt

= - / (ft (t) , X (t)) dt .

a

That (iii) implies (i) follows directly from this identity . To prove that (i) im - plies (iv) we argue indirectly and assume that there exists a point t0 e [0 , 1] such that ft (t0) is not orthogonal to TY (t0) M . Then there exists a vec - tor v0 e TY (t0) M such that (y (t0) , v0) > 0 . We may assume without loss of generality that a <t0 < b . Then there exists a constant e > 0 such that a <t0 — e <t0 + e<b and

t0 — e<t<t0 + e =^ (ft (t) , n (q (t)) v0) > 0 .

Now choose a smooth cutoff function ft : I ^ [0 , 1] such that ft (t) = 0 for all t e I with |t —10| > e and ft (t0) = 1 . Define X e TYQp , q by

X (t) := ft (t) n (y (t)) v0 for t e I .

Then (ft (t) , X (t)) > 0 for all t and (ft (t0) , X (t0)) > 0 . Hence

dE (y) X = — f (ft (t) , X (t)) dt < 0 a

and so 7 does not satisfy (i) . Thus (i) is equivalent to (iii) and (iv) .

We prove that (i) is equivalent to (ii) . Assume first that 7 satisfies (i) . Then 7 also satisfies (iv) and hence 7 (t) T TY (t) M for all t e I . This implies

1 d

0 = <7 (t) , Y (t)) = - ^ |j (t) |2 .

Hence the function I ^ r : t ^ |7 (t) |2 is constant . Choose c > 0 such that |7 (t) | = c . If c = 0 then 7 (t) is constant and so j (t) = p = q . If c > 0 then

[|js (t) | dt

s=0 - J a rb g

|js (t) | dt

s=0

ds

= f|Y (t) rI (drs

= - / (Y (t) , X (t)) dt

c J a * '

= - dE (j) X .

Thus , in the case c > 0 , 7 is an extremal point of E if and only if it is an extremal point of L . Hence (i) is equivalent to (ii) .

We prove that (iii) is equivalent to (v) . Let ($, 7 , 7') be a development of M along M' = rm . Then f (t) = $ (t) i (t) and |$ (t) X (t) = $ (t) VX (t) for all X e Vect (j) and all t e I . Take X = - 7 to obtain 7' (t) = $ (t) Vj (t) for all t e I . Thus Vj = 0 if and only if 7' = 0 . This proves Theorem

Notes

Remark shows that reparametrization by a nundecreasing surjective map a : I' ^ I gives rise to map

Qp , q (I) ^ Qp , q (I') : j ^ Y 0 a which preserves the length functional , i . e .

l (Y 0 a) = l (Y)

for all y e Qp , q (I) . Thus the chain rule in infinite dimensions should assert that if yoa is an extremal (i . e . critical) point of L , then y is an extremal point of L . moreover , if a is a diffeomorphism the map y ^ Y ∘ a is bijective and should give rise to a bijective correspondence between the extremal points of L on Qp , q (I) and those on Qp , q (I') . Finally , if the tangent vector field j vanishes nowhere , then Y can be parametrized by the arclength . This is spelled out in more detail in the next exercise .

Exercise . Let y : I = [a , b] ^ M be a smooth curve such that

7 (t) = 0

for all t <= I and define

T := L (7) = f |^ (t) | dt .

J a

Prove that there exists a unique diffeomorphism a : [0 , T] ^ I such that

a (t') = t   t' = / |'y (s) | ds

a

for all t' <= [0 , T] and all t <= [a , b] . Prove that 7' := 7 o a : [0 , T] ^ M is parametrized by the arclength , i . e . |Y' (t') | = 1 for all t' <= [0 , T] .

Prove that

f b

dL (Y) X = - / (^ (t) , X (t)) dt , V (t) := |^f (t) | - 1 7 (t) . (4 . 1 . 11)

a

Hint: See the relevant formula in the proof of Theorem .

Prove that 7 is an extremal point of L if and only if the curve 7' in part (i) is a geodesic .

Prove that 7 is an extremal point of L if and only if there exists a geodesic 7' : I' ^ M and a diffeomorphism a : I' ^ I such that 7' = 7 o a .

Next we generalize this exercise to cover the case where 7 is allowed to vanish . Recall from Remark that the function s ^ L (qs) need not be differentiable . As an example consider the case where 7 = 70 is constant

Exercise . Let 7 : I ^ M be a smooth curve and let X <= TYQp , q (I) . Choose a smooth curve of curves r ^ Qp , q (I) : s ^ Ys that satisfies Prove that the one - sided derivatives of the function s ^ L (ys) exist at s = 0 and satisfy the inequalities

Exercise . Let (T , y , y') be a development of M along M' . Show that y is a geodesic in M if and only if y' is a geodesic in M' .

Distance

Assume that M c rn is a connected smooth m - dimensional submanifold . Two point p , q e M are of distance \p — q\ apart in the ambient Euclidean space rn . In this section we define a distance function which is more in - timately tied to M by minimizing the length functional over the space of curves in M with fixed endpoints . Thus it may happen that two points in M have a very short distance in rn but can only be joined by very long curves in M . This leads to the intrinsic distance in M . Throughout we denote by I = [0 , 1] the unit interval and , for p , q e M , by

Qp , q := {y : [0 , 1] ^ M \ 7 is smooth and 7 (0) = p , 7 (1) = q}

the space of smooth paths on the unit interval joining p to q . Since M is connected the set Qp , q is nonempty for all p , q e M . (Prove this!)

Definition . The intrinsic distance between two points p , q e M is

the real number d (p , q) > 0 defined by

d (p , q) := inf L (7) .

~y<=Qp , q

The inequality d (p , q) > 0 holds because each curve has nonnegative length and the inequality d (p , q) < holds because Qp , q = 0 .

Remark . Every smooth curve 7 : [0 , 1] ^ rn with endpoints 7 (0) = p and 7 (1) = q satisfies the inequality

Thus d (p , q) >\p — q\ . For 7 (t) := p + t (q — p) we have equality and hence the straight lines minimize the length among all curves from p to q .

Lemma . The function d : M x M ^ [0 , to) defines a metric on M:

If p , q G M satisfy d (p , q) = 0 then p = q .

For all p , q G M we have d (p , q) = d (q , p) .

For all p , q , r G M we have d (p , r) < d (p , q) + d (q , r) .

Proof . By Remark we have d (p , q) > |p — q| for all p , q G M and this proves part (i) . Part (ii) follows from the fact that the curve y (t) := 7 (1 — t) has the same length as 7 and belongs to Qq , p whenever 7 G Hp , q . To prove part (iii) fix a constant e > 0 and choose curves 70 G Hp , q and 71 G Hq , r such that L (7o) < d (p , q) + e and L (y1) < d (q , r) + e . By Remark 4 . 1 . 3 we may assume without loss of generality that 70 (1 — t) = 71 (t) = q for t > 0 sufficiently small . Under this assumption the curve

for 0 < t < 1 / 2 , 1) ,

for 1 / 2 <t < 1

is smooth . Moreover , 7 (0) = p and 7 (1) = r and so 7 G Qp , r . Thus

d (p , r) < L (y) = L (y0) + L (y 1) < d (p , q) + d (q , r) + 2e .

Hence d (p , r) < d (p , q) + d (q , r) + 2e for every e > 0 .

Remark . It is natural to ask if the infimum in is always attained .

This is easily seen not to be the case in general . For example , let M result from the Euclidean space rm by removing a point p0 . Then the distance d (p , q) = |p — q| is equal to the length of the line segment from p to q and any other curve from p to q is longer . Hence if p0 is in the interior of this line segment the infimum is not attained . We shall prove below that the infimum is attained whenever M is complete .

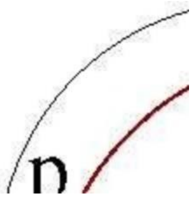

Figure : A geodesic on the 2 - sphere .

Example Let

M := S2 = {p e r3 | |p| = 1}

be the unit sphere in r3 and fix two points p , q e S2 . Then d (p , q) is the length of the shortest curve on the 2 - sphere connecting p and q . Such a curve is a segment on a great circle through p and q and its length is

d (p , q) = cos - 1 ((p , q)) ,

where (p , q) denotes the standard inner product , and we have

0 < d (p , q) < n .

We now have two topologies on our manifold M c rn , namely the topol - ogy determined by the metric d in Lemma and the relative topology inherited from rn . The latter is also determined by a distance function , namely the extrinsic distance function defined as the restriction of the Euclidean distance function on rn to the subset M . We denote it by

do : M x M ^ [0 , to) , d0 (p , q) := |p — q| .

A natural question is if these two metrics d and d0 induce the same topology on M . In other words is a subset U C M open with respect to d0 if and only if it is open with respect to d? Or , equivalently , does a sequence pv e M converge to p0 e M with respect to d if and only if it converges to p0 with respect to d0? Lemma answers this question in the affirmative .

Notes

Exercise . Prove that every translation of rn and every orthogonal transformation preserves the lengths of curves .

Lemma . For every p0 e M we have

lim d<M) =1 .

p , q—po |p — q|

Lemma . Let p0 e M and let 00 : U0 ^ Q0 be a coordinate chart onto an open subset of rm such that its derivative d^0 (p0) : TPoM ^ rm is an orthogonal transformation . Then

lim d (p , q) = 1 PW - mpo |00 (p) — <Mq) | .

The lemmas imply that the topology M inherits as a subset of rm , the topology on M determined by the metric d , and the topology on M induced by the local coordinate systems on M are all the same .

Corollary . For every subset U c M the following are equivalent .

U is open with respect to the metric d in

U is open with respect to the metric d0 in

For every coordinate chart <fo : U0 ^ Q0 of M onto an open sub - set Q0 c rm the set f0 (U0 n U) is an open subset of rm .

Proof . By Remark we have

|p - q\ < d (p , q)

for all p , q G M . Thus the identity idM : (M , d) ^ (M , d0) is Lipschitz continuous with Lipschitz constant one and so every d0 - open subset of M is d - open . Conversely , let U c M be a d - open subset of M and let p0 G U and <= > 0 . Then , by Lemma there exists a constant 5 > 0 such that all p , q G M with \p — p0\ <5 and \q — p0\ <5 satisfy

d (p , q) < (1 + <=) \p — q\ .

Since U is d - open , there exists a constant p > 0 such that

Bp (p0 , d) c U .

With

P0 := min{5' r + e}

this implies BP0 (p0 , d0) c U . Namely , if p G M satisfies

\p — p0\ < P0 < 5

then

d (p , p0) < (1 + <=) \p — p0\ < (1 + <=) P0 < P

and so p G U . Thus U is d0 - open and this proves that (i) is equivalent to (ii) .

That (ii) implies (iii) follows from the fact that each coordinate chart <f0 is a homeomorphism . To prove that (iii) implies (i) , we argue indirectly and assume that U is not d - open . Then there exists a sequence pv G M \ U that converges to an element p0 G U . Let <f0 : U0 ^ Q0 be a coordinate chart with p0 G U0 . Then limv^^\00 (pv) — 00 (p0) \ = 0 by Lemma Thus ^0 (U0 n U) is not open and so U does not satisfy (iii) . □

Proof of Lemma Estimate |p — q\ < d (p , q) holds for all p , q € M . The lemma asserts that , for all po € M and all e > 0 , there exists a do - open neighborhood Uo C M of p0 such that all p , q € Uo satisfy

\p — q\ < d (p , q) < (1 + e) \p — q\ .

Let p0 € M and e > 0 , and define x : rn ^ Tp0M and y : rn ^ Tp0M± by

x (p) :=n (po) (p — po) , y (p) := (1 — n (po)) (p — po) ,

where n (po) : rn ^ Tp0M denotes the orthogonal projection as usual . Then the derivative of the map x\M : M ^ Tp0M at p = po is the iden - tity on Tp0M . Hence the Inverse Function Theorem asserts that the map x\M : M ^ Tp0M is locally invertible near po . Extending this inverse to a smooth map from Tp0 M to rn and composing it with the map y : M ^ Tp0 Mx , we obtain a smooth map

f : Tp0 M ^ Tp0 M^ and an open neighborhood W C rn of po such that

p € M y (p) = f (x (p))

for all p € W Moreover , by definition the map f satisfies

f (0) = 0 € Tp0Mx , df (0) = 0 : Tp0M ^ Tp0Mx Hence there exists a constant 5 > 0 such that , for every x € Tp0M , we have

\x\ < 5 =^ x + f (x) € W and ||df (x) || = sup \df (^) a;\ < e .

o=reTP0 m M

Uo := {p e M n W ||x (p) | <5} .

Given p , q e U0 let y : [0 , 1] ^ M be the curve whose projection to the x - axis is the straight line joining x (p) to x (q) , i . e .

x (Y (t)) = x (p) + t (x (q) — x (p)) =: x (t) ,

y (Y (t)) = f (x (y (t))) = f (x (t)) =: y (t) .

Then y (t) e U0 for all t e [0 , 1] and

l (Y) = f |x (t) + y (t) | dt

Jo

= / |x (t) + df (x (t)) x (t) | dt Jo

[l1 + iidf (x (t)) y) |x (t) | dt oJ

(1 + e) / |;x (t) | dt

o

= (1 + e) |x (p) — x (q) |

= (1 + e) |n (po) (p — q) |

(1 + e) |p — q| .

Hence d (p , q) < L (y) < (1 + e) |p — q|

we have

|d^o (po) v| = |v|

for all v e Tp0M . Fix a constant e > 0 . Then , by continuity of the derivative , there exists a d0 - open neighborhood M0 c M of p0 such that for all p e M0 and all v e TpM we have

(1 — e) |d0o (p) v| < |v| < (1 + e) |d0o (p) v| .

Thus for every curve y : [0 , 1] ^ M0 we have

(1 — e) L (^o o y)) < L (y) < (1 + e) L (^o ◦ y) .

One is tempted to take the infimum over all curves y : [0 , 1] ^ M0 joining two pints p , q e M0 to obtain the inequality

(1 — e) |0o (p) — 0o (q) | < d (p , q) < (1 + e) |0o (p) — 0o (q) | .

However , we must justify these inequalities by showing that the infimum over all curves in Mo agrees with the infimum over all curves in M joining the points p and q .

It suffices to show that the inequalities hold on a smaller heighbor - hood M1 c Mo of po . Choose such a smaller neighborhood M1 such that the open set 0o (M1) is a convex subset of fio . Then the right inequality in follows by taking the curve 7 : [0 , 1] ^ M1 from 7 (0) = p to 7 (1) = q such that 0o o 7 : [0 , 1] ^ 0o (M1) is a straight line . To prove the left inequality we use the fact that M0 is d - open by Lemma . Hence , after shrinking M1 if necessary , there exists a constant r > 0 such that

po e M1 c Br (po , d) c B3r (po , d) c Mo .

Then , for p , q e M1 we have d (p , q) < 2r while L (y) > 4r for any curve 7 from p to q which leaves Mo . Hence the distance d (p , q) of p , q e M1 is the infimum of the lengths L (y) over all curves 7 : [0 , 1] ^ Mo that join 7 (0) = p to 7 (1) = q .

A next question one might ask is the following . Can we choose a coordinate chart 0 : U ^ fi on M with values in an open set fi c rm so that the length of each smooth curve 7 : [0 , 1] ^ U is equal to the length of the curve c := 0 o 7 : [0 , 1] ^ fi? We examine this question by considering the inverse map f := 0 - 1 : fi ^ U . Denote the components of x and f (x) by x = (x1 , . . . , xm) e fi , f (x) = (^1 (x) , . . . , fn (x)) e U .

Given a smooth curve [0 , 1] ^ fi : t ^ c (t) = (c1 (t) , . . . , cm (t)) we can write the length of the composition 7 = f o c : [0 , 1] ^ M in the form

L (f o c) = /

o

n f d \2

E (ddSv (c (t))) dt

V=1 x 7

2

E (E§a (c (t)) ci (t) l dt

dx

v=1 \i=1

A ^ . . v

^ ^ <9x* (c (t)) dxj

v=1 i , j=1

^ (ii (t) gij (c (t)) cj (t) dt .

i , j=1

Here the functions gij : Q ^ r are defined by

g« <x) := E % <x) Ht - <x) = (dX <x) H <x)) . <4 - 2 - 8)

V=1

Thus we have a smooth function g = (gij) : Q ^ rmxm with values in the positive definite matrices given by g (x) = d^ (x) Td^ (x) such that

L ($ o c) = jf \Jc (t) Jg (c (t)) c (t) dt

for every smooth curve c : [0 , 1] ^ Q . Thus the condition L (^ o c) = L (c) for every such curve is equivalent to gij (x) = $ij

for all x <= Q or , equivalently ,

d^ (x) Td^ (x) = H .

This means that ^ preserves angles and areas . The next example shows that for M = S2 it is impossible to find such coordinates .

Example Consider the manifold M = S2 . If there is a diffeomorphism ^ : Q ^ U from an open set Q C r2 onto an open set U C S2 that satisfies it has to map straight lines onto arcs of great circles and it preserves the area . However , the area A of a spherical triangle bounded by three arcs on great circles satisfies the angle sum formula

a + ft + y = n + A .

Hence there can be no such map ^ .

5.3 EXPONENTIAL MAP

Geodesic Spray

The tangent bundle TM is a smooth 2m - dimensional manifold in rn x rn by Corollary . The next lemma characterizes the tangent bundle of the tangent bundle .

Lemma . The tangent space of TM at (p , v) € TM

Proof . We prove the inclusion "c" . Let (p) 7) € T (p , v) TM and choose a smooth curve r ^ TM : t ^ (7 (t) , X (t)) such that

7 (0) = p , X (0) = v , 7 (0) = p , X (0) = 7 .

Then the GauB - Weingarten formula asserts that

X (t) = VX (t) + h7W (y (t) , X (t))

and hence (H — n (y (t))) X (t) = h7 (t) (y (t) , X (t)) for all t € r . Take t = 0 to obtain (1 — n (p)) 7 = hp (p , v) . This proves the inclusion "c" in Equality holds because both sides of the equation are 2m - dimensional linear subspaces of rn x rn . □

By Lemma a smooth map S = (Si , S2) : TM ^ rn x rn is a vector field on TM if and only if

Si (p , v) € TpM , (1 — n (p)) S2 (p , v) = hp (Si (p , v) , v)

for all (p , v) € TM . A special case is where S1 (p , v) = v . Such vector fields correspond to second order differential equations on M .

Definition (Spray) . A vector field S € Vect (TM) is called a spray if it has the form S (p , v) = (v , S2 (p , v)) where S2 : TM ^ rn is a smooth map satisfying

(1 — n (p)) S2 (p , v) = hp (v , v) , S2 (p , Av) = A2S2 (p , v) for all (p , v) € TM and A € r . The vector field S € Vect (TM) defined by

S (p , v) := (v , hp (v , v)) € T (p , v) TM for p € M and v € TpM is called the geodesic spray .

Exponential Map

Lemma Let y : I ^ M be a smooth curve on an open interval I C r . Then 7 is a geodesic if and only if the curve I ^ TM : t ^ (Y (t) , Y (t)) is an integral curve of the geodesic spray S .

Proof . A smooth curve I ^ TM : t ^ (7 (t) , X (t)) is an integral curve of S if and only if

Y (t) = X (t) ^ (t) = hY (t) (X (t) , X (t))

for all t <= I . By equation , this holds if and only if 7 is a geodesic and Y = X .

Lemma Let M C rn be an m - dimensional submanifold .

(i) For every p <= M and every v <= TpM there is an e > 0 and a smooth

curve y : (—e , e) M such that

VY = 0 , y (0) = p , Y (0) = v .

(il) If Yi : Ii ^ M and Y2 : I2 ^ M are geodesics and t0 <= Ii C I2 with

Yi (to) = Y2 (to) , - y i (to) = Y 2 (to) then Yi (t) = Y2 (t) for all t <= Ii C I2 .

Definition (Exponential Map) . For p <= M and v <= TpM the interval

I 11 Ii r I is an open interval containing 0 and there is a 1

p , v U* | geodesic y : I ^ M satisfying y (0) = p , 7 (0) = v J .

is called the maximal existence interval for the geodesic through p in the direction v . For p <= M define the set Vp C TpM by

Vp := {v <= TPM | 1 <= IP}V} .

The exponential map at p is the map

expp : Vp ^ M

that assigns to every tangent vector v <= Vp the point expp (v) := y (1) , where y : Ip , v ^ M is the unique geodesic satisfying y (0) = p and Y (0) = v .

Lemma . (i) The set

V := { (p , v) | p e M , v e Vp} C TM

is open and the map V ^ M : (p , v) ^ expp (v) is smooth .

(ii) If P e M and v e Vp , then

Ip , v = {t e r | tv e Vp}

and the geodesic y : Ip , v ^ M with 7 (0) = p and 7 (0) = v is given by

Y (t) = expp (tv) , t e Ip , v .

Proof . Part (i) follows directly from Lemma .

To prove part (ii) , fix an element p e M and a tangent vector v e Vp , and let 7 : Ip , v ^ M be the unique geodesic with 7 (0) = p and 7 (0) =

Notes

v . Fix a nonzero real number A and define the map Ya : A - 1Ip , v ^ M by

7a (t) := 7 (At) for t e A - 1Ip , v .

Then YA (t) = AY (At) ans YA (t) = A2Y (At) and hence

VYA (t) = n (YA (t)) YA (t) = A2n (Y (At)) Y (At) = A2Vy (A t) = 0

for every t e A - 1Ip , v . This shows that ya is a geodesic with

Ya (0) = p , Ya (0) = Av .

In particular , we have A - 1Ip , v C Ip , Av . Interchanging the roles of v and Av we obtain A - 1Ip , v = Ip , Av . Thus

A e Ip , v 1 e Ip , Av Av e Vp

and

Y (A) = Ya (1) = expp (Av) for A e Ip , v . □

Since expp (0) = p by definition , the derivative of the exponential map at v = 0 is a linear map from TpM to itself . This derivative is the identity map as illustrated and proved in the following corollary .

Corollary . The map expp : Vp ^ M is smooth and its derivative at the origin is dexpp (0) = id : TpM ^ TpM .

Proof . The set Vp is an open subset of the linear subspace TpM c rn , with respect to the relative topology , and hence is a manifold . The tangent space of Vp at each point is TpM the exponential map expp : Vp ^ M is smooth and its derivative at the origin is given by

expp (tv) = 7 (0) = ^

t=0

where 7 : Ipv ^ M is once again the unique geodesic through p in the direction v .

Corollary . Let p <= M and , for r > 0 , denote

Br (p) := {v <= TpM | |v| < r} .

If r > 0 is sufficiently small then Br (p) c Vp , the set

Ur (p) := expp (Br (p))

is an open subset of M , and the restriction of the exponential map to Br (p) is a diffeomorphism from Br (p) to Ur (p) .

Definition (Injectivity Radius) . Let M c rn be a smooth mmanifold . The injectivity radius of M at p is the supremum of all r > 0 such that the restriction of the exponential map expp to Br (p) is a diffeomorphism onto its image

Ur (p) := expp (Br (p)) .

It will be denoted by

inj (p) := inj (p; M) := sup < r > 0

The injectivity radius of M is the infimum of the injectivity radii of M at p over all p <= M . It will be denoted by

inj (M) := inf inj (p; M) .

pGM

Example The exponential map on rm is given by

expp (v) = p + v for p , v <= rm .

For every p <= rm this map is a diffeomorphism from TpRm = rm to rm and hence the injectivity radius of rm is infinity .

Example The exponential map on Sm is given by

sin (|v|)

expp (v) = cos (|v|) p +

for every p <= Sm and every nonzero tangent vector v <= TpSm = p . The re - striction of this map to the open ball of radius r in TpM is a diffeomorphism onto its image if and only if r < n . Hence the injectivity radius of Sm at every point is n . Exercise: Given p <= Sm and 0 = v <=

Notes

TpSm = p± , prove that the geodesic 7 : r ^ Sm with 7 (0) = p and 7 (0) = v is given by Y (t) = cos (t |v|) p + sl"|VjV) v for t <= r . Show that , in the case 0 < |v| < n there is no shorter curve in Sm connecting p and q and deduce that the intrinsic distance on Sm is given by d (p , q) = cos - 1 ((p , q)) for p , q <= Sm

Example . Consider the orthogonal group O (n) C rnxn with the standard inner product

{v , w) := trace ^vTwj

on rnxn . The orthogonal projection n (g) : rnxn ^ TgO (n) is given by

1

2

and the second fundamental form by

kg (v , v) = —gvTv .

Hence a curve y : r ^ O (n) is a geodesic if and only if yt7 + YJY = 0 or , equivalently , yTY is constant . This shows that geodesics in O (n) have the form Y (t) = gexp (t{) for g <= O (n) and { <= o (n) . It follows that the exponential map is given by

expg (v) = g exp (g - 1v) = exp (vg - 1) g

for g <= O (n) and v <= TgO (n) . In particular , for g = H the exponential map expj : o (n) ^ O (n) agrees with the exponential matrix .

Exercise . What is the injectivity radius of the 2 - torus t2 = S1 x S1 , the punctured 2 - plane r2 \ { (0 , 0) } , and the orthogonal group O (n) ?

Convex Neighborhoods

Geodesics in Local Coordinates

Lemma . Let M c rn be an m - dimensional manifold and choose a coordinate chart 0 : U ^ Q with inverse

0 := 0 - 1 : Q ^ U .

Let rk : Q ^ r be the Christoffel symbols defined by and let c : I ^ Q be a smooth curve . Then the curve 7 := 0 o c : I ^ M is a geodesic if and only if c satisfies the 2nd order differential equation

m

ck + <= rj (c) cic> = 0

i , j=1

for k = 1 , . . . , m .

Proof . This follows immediately from the definition of Geodesics and equation in Lemma with X = 7 and { = c . □

That is based on the existence and uniqueness of solutions of second order differential equations in local coordinates .

Exercise . Let Q c rm be an open set and g = (gj) : Q ^ rm*m be a smooth map with values in the space of positive definite symmetric matrices . Consider the energy functional

E (c) := [L (c (t) , c (t)) dt J 0

on the space of paths c : [0 , 1] ^ Q , where L : Q x rm ^ r is defined by

m

L (x , <=) := - <= Cigij (x) e .

i , j=1

The Euler—Lagrange equations of this variational problem have the form

d dL . . . . . . dL , , , , , , , . .

Jtd^k (c (t) , c (t)) = djXk (c (t) , c (t)) , k = 1 , . . . , m .

Prove that the Euler - Lagrange equations are equivalent to the geodesic equations , where the rk : Q ^ r are given

Check your Progress - 1

Discuss Geodesics

Discuss Exponential Map

5.4 LET US SUM UP

In this unit we have discussed the definition and example of Geodesics , Exponential Map

5.5 KEYWORDS

Geodesics ….. **T**his chapter introduces Geodesics in Riemannian manifolds

Exponential Map…… Geodesic Spray..The tangent bundle TM is a smooth 2m - dimensional manifold in rn x rn

5.6 QUESTIONS FOR REVIEW

Explain Geodesics

Explain Exponential Map

5.7 ANSWERS TO CHECK YOUR PROGRESS

Geodesics (answer for Check your Progress - 1 Q)

Exponential Map (answer for Check your Progress - 1 Q)

5.8 REFERENCE

Differential Geometry, Differential Geometry & Application,
Introduction to Defferential Geometry, Basic of Differential Geometry.

UNIT-VI: CONVEXITY

STRUCTURE

6.0 Objectives

6.1 Introduction

6.2 Convexity

6.3 Minimal Geodesics

6.4 Completeness and Hopf—Rinow

6.5 Geodesics in the Intrinsic Setting…..Intrinsic Distance

6.6 Let Us Sum Up

6.7 Keywords

6.8 Questions For Review

6.9 Answers To Check Your Progress

6.10 References

6.0 OBJECTIVES

After studying this unit , you should be able to:

- Understand about Convexity
- Minimal Geodesics
- Completeness and Hopf—Rinow
- Geodesics in the Intrinsic Setting….. Intrinsic Distance

6.1 INTRODUCTION

Differential geometry arose and developed as a result of and in connection to the mathematical analysis of curves and surfaces Mathematical analysis of curves and surfaces had been developed to answer some of unanswered questions that appeared in calculus like the reasons for relationships between complex shapes and curves , series and analytic functions Convexity , Minimal Geodesics , Completeness and Hopf—Rinow, Geodesics in the Intrinsic Setting…..Intrinsic Distance

6.2 CONVEXITY

A subset of an affine space is called convex iff it contains the line segment joining any two of its points . The definition carries over to a submanifold M of Euclidean space (or indeed more generally to any manifold M equipped with a spray) once we reword the definition so as to confront the difficulty that a geodesic joining two points might not exist nor , if it does , need it be unique .

Definition (Geodesically Convex Sets) . Let M C rn be a smooth m - dimensional manifold . A subset U C M is called geodesically con - vex if , for all po , Pi G U , there exists a unique geodesic 7 : [0 , 1] ^ U such that 7 (0) = p0 and 7 (1) = p1 .

It is not precluded in Definition that there be other geodesics from p to q which leave and then re - enter U , and these may even be shorter than the geodesic in U .

Exercise . (a) Find a geodesically convex set U in a manifold M and points po , pi G U such that the unique geodesic 7 : [0 , 1] ^ U with 7 (0) = p0 and 7 (1) = p1 has length L (y) > d (p0 , p1) . Hint: An interval of length bigger than n in S1 .

(b) Find a set U in a manifold M such that any two points in U can be connected by a minimal geodesic in U , but U is not geodesically convex . Hint: A closed hemisphere in S2 .

Theorem (Convex Neighborhoods) . Let M C rn be a smooth m - dimensional submanifold and fix a point p0 G M . Let $: U ^ Q be any coordinate chart on an open neighborhood U C M of p0 with values in an open set Q C rm . Then the set

Ur := {p G U | |$ (p) - 0 (p0) | < r}

is geodesically convex for r > 0 sufficiently small . Corollary Let M C rn be a smooth m - manifold and let p0 G M . Then , for r > 0 sufficiently small , the open ball

Ur (P0) := {p G M | d (p0 , p) < r}

is geodesically convex .

Proof . Choose an orthonormal basis ei , . . . , em of Tp0 M and define

Q := {x € rm | \x\ < inj (p0; M) } ,

U := {p € M \ d (po , p) < inj (po; M) } .

Define the map f : Q ^ U byf (x) := exppo (E x'eA

for x = (x1 , . . . , xm) € Q . Then f is a diffeomorphism and d (p0 , f (x)) = \x\ for all x € Q . Hence its inverse

f := f - 1 : U ^ Q

satisfies f (p0) = 0 and \f (p) \ = d (p0 , p) for all p € U . Thus

Ur (po) = {p € U \ \f (p) - f (po) \ < r} for 0 < r < inj (po; M)

Definition (Geodesically Normal Coordinates) . The coordinate chart f : U ^ Q in sends Geodesics through p0 to straight lines through the origin . Its components x1 , . . . , xm : U ^ r are called geodesically normal coordinates at p0 .

Proof of Theorem . Assume without loss of generality that f (p0) = 0 . Let rkj : Q ^ r be the Christoffel symbols of the coordinate chart and , for x € Q , define the quadratic function Qx : rm ^ r by

m ^ m

. M xkrfc

Jj 1 . ij \

Qx (e) := E (fk) - E xkrkj (x) eiej .

k=1 i , j , k=1

Shrinking U , if necessary , we may assume that

m

E xk rkj (x)

< - — for all x G Q .

2m

k= 1

Then , for all x € Q and all f € rm we have

1 (m \2 1

Qx<« f \ 2 - 2m ^ 2 \f \ 2 ^ 0 -

Hence Qx is positive definite for every x € Q .

Now let y : [0 , 1] ^ U be a geodesic and define

c (t) := 0 (y (t))

for 0 < t < 1 . Then , by Lemma , c satisfies the differential equation

ck + ^ rj (c) cicj = 0 .

i , j

Hence

d2 |c|2 d , . l2 ^ . . |c|2

dt* IT = dt <c'c> = |c| + <c'c> = Qc (c) - "iT - 0

and so the function t ^ |^ (Y (t)) |2 is convex . Thus , if y (0) , Y (1) <= Ur for some r > 0 , it follows that y (t) <= Ur for all t <= [0 , 1] .

Consider the exponential map

V = { (p , v) <= TM | v <= Vp} ^ M : (p , v) ^ expp (v)

in Lemma . Its domain V is open and the exponential map is smooth . Since it sends the pair (p0 , 0) <= V to exppo (0) = po <= U , it follows from con - tinuity that there exist constants e > 0 and r > 0 such that

p <= Ur , v <= TpM , |v| < e =^ v <= Vp , expp (v) <= U .

Moreover , we have

d expp0 (0) = id : TpoM ^ Tpo M

Notes

Hence the Implicit Function Theorem asserts that the constants e > 0 and r > 0 can be chosen such that holds and there exists a smooth map h : Ur x Ur ^ rn that satisfies the conditions

h (p , q) <= TpM , |h (p , q) | <e

for all p , q <= Ur and

expp (v) = q v = h (p , q)

for all p , q <= Ur and all v <= TpM with |v| < e . In particular , we have

h (po , po) = 0 and expp (h (p , q)) = q for all p , q <= Ur .

Fix two constants e > 0 and r > 0 and a smooth map h : Ur x Ur ^ rn are satisfied . We show that any two points p , q <= Ur are joined by a geodesic in Ur .

Let p , q <= Ur and define

Y (t) := expp (th (p , q)) for 0 < t < 1 .

This curve y : [0 , 1] ^ M is well defined by it is a geodesic satisfying y (0) = p <= Ur by Lemma , it satisfies y (1) = q <= Ur it takes values in U by and so y ([0 , 1]) C Ur because the function [0 , 1] ^ r : t ^ |^ (Y (t)) |2 is convex .

We show that there exists at most one geodesic in Ur joining p and q . Let p , q <= Ur and let y : [0 , 1] ^ Ur be any geodesic such that y (0) = p and y (1) = q . Define v := Y (t) <= TpM . Then y (t) = expp (tv) for 0 < t < 1 by Lemma We claim that |v| < e . Suppose , by contradiction , that

|v| > e .

Then

T := ^ < 1 | v|

and , for 0 < t < T , we have |tv| < e and expp (tv) = y (t) <= Ur and so

h (p , Y (t)) = tv .

by . Thus

|h (p , Y (t)) | = t|v| for 0 < t < T .

Take the limit t ^ T to obtain

|h (p , y (T)) | = T |v| = e in contradiction . This contradiction shows that |v| < e . Since

expp (v) = y (1) = q <= Ur it follows from that v = h (p , q) . This proves Theorem□

Exercise . Consider the set Ur (p) = {q <= M | d (p , q) < r} for p <= M and r > 0 . Corollary asserts that this set is geodesically convex for r sufficiently small . How large can you choose r in the cases

M = S2 , M = t2 = S1 x S1 , M = r2 , M = r2 \ {0} .

Compare this with the injectivity radius . If the set Ur (p) in these exam - ples is geodesically convex , does it follow that every geodesic in Ur (p) is minimizing?

6.3 MINIMAL GEODESICS

Any straight line segment in Euclidean space is the shortest curve joining its endpoints . The analogous assertion for geodesics in a manifold M is false; consider for example an arc which is more than half of a great circle on a sphere . In this section we consider curves which realize the shortest distance between their endpoints .

Lemma . Let I = [a , b] be a compact interval , let 7 : I ^ M be a smooth curve , and define p := 7 (a) and q := 7 (b) . Then the following are equivalent .

7 is parametrized proportional to the arclength , i . e . |7 (t) | = c is constant , and y minimizes the length , i . e .

l (y) < l (y0

for every smooth curve Y in M joining p and q .

y minimizes the energy , i . e .

E (y) < E (V)

for every smooth curve Y : I ^ M with y' (a) = p and y' (b) = q .

Proof . See page 192 . □

Definition (Minimal Geodesic) . A smooth curve y : I ^ M on

a compact interval I C r is called a minimal geodesic if it satisfies the equivalent conditions of Lemma

Remark . (i) Condition (i) says that (the velocity |Y| is constant and) L (y) = d (p , q) , i . e . that y is a shortest curve from p to q . It is not precluded that there be more than one such y; consider for example the case where M is a sphere and p and q are antipodal .

(ii) Condition (ii) implies that

e (Y s) = 0

s=0

for every smooth variation r x I ^ M : s ^ Ys (t) of Y with fixed endpoints . Hence a minimal geodesic is a geodesic .

(iii) Finally , we remark that L (y) (but not E (y)) is independent of the parametrization of y . Hence if y is a minimal geodesic L (y) < L (y') for every y' (from p to q) whereas E (y) < E (y') for those Y defined on (an interval the same length as) I .

Proof of Lemma . We prove that (i) implies (ii) . Let (c) be the (con - stant) value of |7 (t) | . Then

l (y) = (b - a) c e (y) = (b 2°) c .

Then , for every smooth curve 7' : I ^ M with 7' (a) = p and 7' (b) = q , we have

4E (7) 2 = c2l (7) 2

c2l (7') 2

= c2 (I '^' (t) ' dt)

/ "b

c2 (b - a) '7 / (t)

J a

= 2 (b — a) c2E (7')

= 4E (7) E (7') .

Here the fourth step follows from the Cauchy - Schwarz inequality . Now divide by 4E (7) to obtain E (7) < E (7') .

We prove that (ii) implies (i) . We have already shown in Remark that (ii) implies that 7 is a geodesic . It is easy to dispose of the case where M is one - dimensional . In that case any 7 minimizing E (7) or L (7) must be monotonic onto a subarc; otherwise it could be altered so as to make the integral smaller . Hence suppose M is of dimension at least two . Suppose , by contradiction , that L (7') < L (7) for some curve 7' from p to q . Since the dimension of M is bigger than one , we may approximate 7' by a curve whose tangent vector nowhere vanishes , i . e . we may assume without loss of generality that 7' (t) = 0 for all t . Then we can reparametrize 7' proportional to arclength without changing its length , and by a further transformation we can make its domain equal to I . Thus we may assume without loss of generality that 7' : I ^ M is a smooth curve with 7' (a) = p and 7' (b) = q such that |7' (t) | = c' and

(b — a) c' = L (7') < L (7) = (b — a) c .

This implies c' < c and hence

e (y /) = <Y—aT < TWd = e (7) .

This contradicts (ii) and proves Lemma

The next theorem asserts the existence of minimal Geodesics .

Theorem (Existence of Minimal Geodesics) . Let M C rn be a

smooth m - manifold , fix a point p e M , and let r > 0 be smaller than the injectivity radius of M at p . Let v e TpM such that |v| < r . Then

d (p , q) = |v| , q :=expp (v) ,

Notes

and a curve y e Qp , q has minimal length L (7) = |v| if and only if there is a smooth map ft : [0 , 1] ^ [0 , 1] satisfying

<= (0) = 0 , ft (1) = 1 , ft > 0

such that y (t) = expp (ft (t) v) for 0 < t < 1 .

Ur

Lemma (Gaufi Lemma) . Let M , p , r be as in Theorem let I C r be an open interval , and let w : I ^ Vp be a smooth curve whose norm

Proof of Lemma . For every t e I we have

a (0 , t) = expp (0) = p and so the assertion holds for s = 0 , i . e .

da . . da , . ,

as (M) 'aS (0 - (h=0 .

Moreover , each curve s ^ a (s , t) is a geodesic , i . e .

^ da rr , . d2a

v>dS = n (a) s 0 .

By Theorem the function

da ,

ds (s - t)

is constant for every t , so that

= |w (t) | = r for (s , t) e r x I .

This implies

da da\ Ida da s ds ' dt / + \ ds , s dt

da TT , , d2a \

as'n (a) aiat /

/ da d2a \

(n (a) as - asa)

da

d2a \ ds' dsdt / d_

dt 0 .

Since the function (, d) vanishes for s = 0 we obtain

da . . da . . ,

ds (M) > =

for all s and t . This proves Lemma Proof of Theorem 4 . 5 . 4 . Let r > 0 be as in Corollary and let v e TpM such that 0 < |v| =: e < r . Denote q := expp (v) and let 7 e Qp , q . Assume first that

Y (t) e expp {Be (p)) = Us V t e [0 , 1] .

Then there is a unique smooth function [0 , 1] ^ TpM : t ^ v (t) such that |v (t) | < e and q (t) = expp (v (t)) for every t . The set

I := {t e [0 , 1] 1 Y (t) = p} = {t e [0 , 1] 1 v (t) = 0} C (0 , 1]

is open in the relative topology of (0 , 1] . Thus I is a union of open intervals in (0 , 1) and one half open interval containing 1 . Define P : [0 , 1] ^ [0 , 1] and w : I ^ TpM by

|v (t) |

w (t) := e

e - - |v (t) |"

Then P is continuous , both P and w are smooth on I ,

P (0) = 0 , P (1) = 1 , w (1) = v ,

and

|w (t) | = e , y (t) = expp (P (t) w (t))

for all t e I . We prove that L (y) > e . To see this let a : [0 , 1] x I ^ M be the map of Lemma , i . e .

a (s , t) := expp (sw (t)) .

Notes

Then Y (t) = a (P (t) , t) and hence

^ (t) =P (t) d? (P (t) , t) + fff (P (t) , t)

for every t > 0 . Hence it follows from Lemma that

for every t e I . Hence

L (y) = [|Y (t) | dt = / |'y (t) | dt > e / P (t) dt > e / P (t) dt = e . do
JI JI Ji

Here the last equation follows by applying the fundamental theorem of cal - culus to each interval in I and using the fact that ft (0) = 0 and ft (1) = 1 . If L (y) = e we must have

o

(ft (t) , t) = 0 , ft (t) > 0 for all t g I .

Thus I is a single half open interval containing 1 and on this interval the condition |a (ft (t) , t) = 0 implies W (t) = 0 . Since w (1) = v we have w (t) = v for every t g I . Hence y (t) = expp (ft (t) v) for every t g [0 , 1] . It follows that ft is smooth on the closed interval [0 , 1] (and not just on I) . Thus we have proved that every 7 g Hp , q with values in U<= has length L (7) > e with equality if and only if 7 is a reparametrized geodesic . But if 7 g Hp , q does not take values only in U<= , there must be a T G (0 , 1) such that y ([0 , T]) c U<= and y (T) g dU<= . Then L (y|[o , T]) > e , by what we have just proved , and L (y|[t1]) > 0 because the restriction of 7 to [T , 1] cannot be constant; so in this case we have L (y) > e . □

The next corollary gives a partial answer to our problem of finding length minimizing curves . It asserts that geodesics minimize the length locally .

Corollary . Let M C rn be a smooth m - manifold , let I C r be an open interval , and let 7 : I ^ M be a geodesic . Fix a point t0 G I . Then there exists a constant e > 0 such that

to - e < s <t < to + e =^ L (y|m) = d (Y (s) , Y (t)) .

Proof . Since 7 is a geodesic its derivative has constant norm |ft (t) | = c Choose 5 > 0 so small that the interval [t0 — 5 , t0 + 5] is contained in I . Then there is a constant r > 0 such that r < inj (q (t)) whenever |t — t0| < 5 . Choose e > 0 such that

e < 5 , 2ec < r .

If t0 — e<s<t<t0 + e then

Y (t) = exp7 (s) ((t — s) Y (s))

and

| (t — s) 'y (s) | = |t — s| c < 2ec < r < inj (Y (s)) .

Hence it follows from Theorem that

l (YIm) = |t — s| c = d (Y (s) , Y (t)) .

This proves Corollary

Exercise How large can the constant e in Corollary be chosen in the case M = S2? Compare this with the injectivity radius .

Remark . We conclude from Theorem that

Sr (p) := {q e M | d (p , q) = r} = expp ({v e TpM | |v| = r}) (4 . 5 . 1)

for 0 < r < inj (p; M) . The GauB Lemma shows that the geodesic rays [0 , 1] ^ M : s ^ expp (sv) emanating from p are the orthogonal tra - jectories to the concentric spheres Sr (p) .

Exercise . Let

M C r3

be of dimension two and suppose that M is invariant under the (orthogonal) reflection about some plane E C r3 . Show that E intersects M in a geodesic . (Hint: Otherwise there would be points p , q e M very close to one an - other joined by two distinct minimal geodesics .) Conclude for example that the coordinate planes intersect the ellipsoid (x / a) 2 + (y / b) 2 + (z / c) 2 = 1 in geodesics .

Exercise . Choose geodesic normal coordinates near p e M via

Notes

where ei , . . . , em is an orthonormal basis of TpM . Then we have xl (p) = 0 and

for 0 < r < inj (p; M) . Hence Theorem asserts that Br (p) is convex for r > 0 sufficiently small .

Show that it can happen that a geodesic in Br (p) is not minimal . Hint: Take M to be the hemisphere { (x , y , z) e r3 | x2 + y2 + z2 = 1 , z > 0} to - gether with the disc { (x , y , z) e r3 | x2 + y2 < 1 , z = 0} , but smooth the corners along the circle x2 + y2 = 1 , z = 0 . Take p = (0 , 0 , 1) and r = n / 2 .

Show that , if r > 0 is sufficiently small , then the unique geodesic 7 in Br (p) joining two points q , q' e Br (p) is minimal and that in fact any curve y' from q to q' which is not a reparametrization of 7 is strictly longer , i . e . L (y') > L (y) = d (q , q') .

Exercise . Let 7 : I = [a , b] ^ M be a smooth curve with end - points y (a) = p and 7 (b) = q and nowhere vanishing derivative , i . e . 7 (t) = 0 for all t <= I . Prove that the following are equivalent .

(i) The curve 7 is an extremal of the length functional , i . e . every

smooth map r x I ^ M : (s , t) ^ 7s (t) with 7s (a) = p and 7s (b) = q for

all s satisfies .

d

= 0 .

s=0

The curve 7 is a reparametrized geodesic , i . e . there exists a smooth map a : [a , b] ^ [0 , 1] with a (a) = 0 , a (b) = 1 , a (t) > 0 for all t <= I , and a vector v <= TpM such that

q = expp (v) 7 (t) = expp (a (t) v)

for all t <= I . (We remark that the hypothesis 7 (t) = 0 implies that a is actually a diffeomorphism , i . e . a (t) > 0 for all t <= I .)

The curve y minimizes the length functional locally , i . e . there exists an e > 0 such that L (7|[s , t]) = d (7 (s) , 7 (t)) for every closed subinterval [s , t] C I of length t — s < e .

It is often convenient to consider curves 7 where 7 (t) is allowed to vanish for some values of t; then 7 cannot (in general) be parametrized by arclength . Such a curve 7 : I ^ M can be smooth (as a map) and yet its image may have corners (where 7 necessarily vanishes) . Note that a curve with corners can never minimize the distance , even locally .

Exercise . The assumption that 7 is nowhere vanishing . Deduce that , if y : I ^ M is a shortest curve joining p to q , i . e . L (y) = d (p , q) , then y is a reparametrized geodesic .

Show by example that one can have a variation (7s}seR of a reparame - trized geodesic y0 = 7 for which the map s ^ L (ys) is not even differentiable at s = 0 .

Show , however , that conditions remain equivalent if the hypothesis that 7 is nowhere vanishing is weakened to the hypothesis that 7 (t) = 0 for all but finitely many t <= I . Conclude that a broken geodesic is a reparametrized geodesic if and only if it minimizes arclength locally . (A broken geodesic is a continuous map 7 : I = [a , b] ^ M for which there exist a = t0 < ti < ■ ■ ■ < tn = b such that 7\[ti - 1 , ti] is a geodesic for i = 1 , . . . , n . It is thus a geodesic if and only if 7 is continuous at the break points , i . e . 7 (t -) = 7 (t +) for i = 1 , . . . , n — 1 .)

6.4 COMPLETENESS AND HOPF—RINOW

For a Riemannian manifold there are different notions of completeness . First , in completeness was defined in terms of the completeness of time dependent basic vector fields on the frame bundle Second there is a distance function

d : M x M ^ [0 , to)

defined by equation so that we can speak of completeness of the metric space (M , d) in the sense that every Cauchy sequence converges . Third

Notes

, there is the question of whether geodesics through any point in any direction exist for all time; if so we call a Riemannian manifold geodesically complete . The remarkable fact is that these three rather different notions of completeness are actually equivalent and that , in the complete case , any two points in M can be joined by a shortest geodesic . This is the content of the Hopf - Rinow theorem . We will spell out the details of the proof for embedded manifolds and leave it to the reader (as a straight forward exercise) to extend the proof to the intrinsic setting .

Definition Let M c rn be an m - dimensional manifold . Given a point p <= M we say that M is geodesically complete at p if , for every tangent vector v <= TpM , there exists a geodesic 7 : r ^ M (on the entire real axis) satisfying 7 (0) = p and 7 (0) = v (or equivalently Vp = TpM where Vp c TpM is defined) . The manifold M is called geodesically complete if it is geodesically complete at every point p <= M .

Definition . Let (M , d) be a metric space . A subset A c M is called bounded if

sup d (p , p0) < to

peA

for some (and hence every) point p0 <= M .

Example A manifold M c rn can be contained in a bounded subset of rn and still not be bounded with respect to the metric . An example is the 1 - manifold

M = { (x , y) <= r2 | 0 < x < 1 , y = sin (1 / x) } .

Exercise . Let (M , d) be a metric space . Prove that every compact subset K c M is closed and bounded . Find an example of a metric space that contains a closed and bounded subset that is not compact .

Theorem (Completeness) . Let M C rn be a connected m - dimensional manifold and let d : M x M ^ [0 , to) be the distance function defined Then the following are equivalent .

M is geodesically complete .

There exists a point p e M such that M is geodesically complete at p .

Every closed and bounded subset of M is compact .

(M , d) is a complete metric space .

M is complete , i . e . for every smooth curve f : r ^ rm and every ele - ment (po , eo) e F (M) there exists a smooth curve ft : r ^ F (M) satisfying

ft (t) = Bm (P (t)) , ft (0) = (Po , eo) .

The basic vector field Bg e Vect (F (M)) is complete for every f e rm .

For every smooth curve Y : r ^ rm , every p0 e M , and every or - thogonal isomorphism $0 : Tp0M ^ rm there exists a development ($, 7 , 7') of M along rm on all of r that satisfies 7 (0) = p0 and $ (0) = $0

Theorem (Hopf—Rinow) . Let M C rn be a connected m - manifold and let p e M . Assume M is geodesically complete at p . Then , for ev - ery q e M , there exists a geodesic 7 : [0 , 1] ^ M such that

7 (0) = p , 7 (1) = q , L (y) = d (p , q) .

We prove that (ii) implies (iii) . Thus assume that M is geodesically complete at the point p0 e M and let K C M be a closed and bounded subset . Then r := supqeKd (p0 , q) < to . Hence Theorem asserts that , for every q e K , there exists a vector v e Tp0M such that |v| = d (p0 , q) < r and expp0 (v) = q . Thus

K C expp0 (Br (p0)) , Br (p0) = {v e TPoM | |v| < r} .

Then B := {v e TP0M | |v| < r , exppo (v) e K} is a closed and bounded sub - set of the Euclidean space TPoM . Hence B is compact and K = exppo (B) . Since the exponential map exppo : TP0 M ^ M is continuous it follows that K is compact . This shows that (ii) implies (iii) .

We prove that (iii) implies (iv) . Thus assume that every closed and bounded subset of M is compact and choose a Cauchy sequence pi e M . Choose io e n such that d (pi , pj) < 1 for all i , j e n with i , j > io . Define c := max d (»i , pi) + 1 .

1<i<io

Then d (pi , pi) < d (pi , pi0) + d (pi0 , pi) < d (pi , pi0) + 1 < c for all i > io and so d (p1 , pi) < c for all i e n . Hence the set {pi | i e n} is bounded and so is its closure . By (iii) this implies that the sequence pi has a convergent subsequence . Since pi is a Cauchy sequence , this implies that pi converges . Thus we have proved that (iii) implies (iv) .

We prove that (iv) implies (v) . Fix a smooth curve <= : r ^ rm and an element (p0 , e0) e F (M) . Assume , by contradiction , that there exists a real number T > 0 such that there exists a solution ft : [0 , T) ^ F (M) of equation that cannot be extended to the interval [0 , T + e) for any e > 0 . Write ft (t) =: (7 (t) , e (t)) so that 7 and e satisfy the equations

7 (t) = e (t) <= (t) e (t) = hY (t) (, T (t)) e (t) , 7 (0) = Po , e (0) = eo .

This implies e (t) n e TY (t) M and e (t) n e T^M for all n e rm and therefore dd

s (n . e (t) Te (00 = dt (e (On . e (t) C) = (e (t) n . e (t) ( ) + (e (f) n . e (t) 0 = 0

for all n , Z e rm and all t e [0 , T) . Thus the function t ^ e (t) Te (t) is con - stant , hence

e (t) Te (t) = e ([eo , ||e (t) y = sup = ||eoH

o=n^Rm |n|

for 0 < t < T , hence

|Y (t) | = le (t) <= (t) l < |eoH l<= (t) l < |eoH sup l<= (s) l =: cT

o <s<T

and so d (q (s) , 7 (t)) < L (y|[s , t]) < (t — s) cT for 0 < s < t < T . Since (M , d) is a complete metric space , this shows that the limit p1 := limt^T q (t) e M exists . Thus the set K := 7 ([0 , T)) U {p1} C M is compact and so is the set

K :^ (p , e) e F (M) | p e K , eTe = eTe^ C F (M) .

By equation (4 . 6 . 2) the curve [0 , T) ^ r x F (M) : t ^ (t , 7 (t) , e (t)) takes values in the compact set [0 , T] x K and is the integral curve of a vector field on the manifold r x F (M) . Hence Corollary asserts that [0 , T) cannot be the maximal existence interval of this integral curve , a contradiction . This shows that (iv) implies (v) .

That (v) implies (vi) follows by taking <= (t) = <= in (v) .

We prove that (vi) implies (i) . Fix an element p0 <= M and a tan - gent vector v0 <= Tp0M . Let eo <= Liso (rm , Tp0M) be any isomorphism and choose <= <= rm such that e0<= = v0 . By (vi) the vector field Bg has a unique integral curve r ^ F (M) : t ^ P (t) = (j (t) , e (t)) with

<= (0) = (po , eo) .

Thus

7 (t) = e (t) <= e (t) = hY (t) (e (t) <=) e (t) ,

and hence

7 (t) = e (t) <= = hY (t) (e (t) <=) e (t) <= = hY« (7 (t) , 7 (t)) .

By the Gaufi - Weingarten formula , this implies Vy (t) = 0 for every t and hence 7 : r ^ M is a geodesic with 7 (0) = p0 and 7 (0) = e0<= = v0 . Thus M is geodesically complete and this shows that (vi) implies (i) .

Lemma . Let M c rn be a connected m - manifold and p <= M . Sup - pose e > 0 is smaller than the injectivity radius of M at p and denote

<=i (p) := {v <= TPM | |v| = 1} , S<= (p) := {p' <= M | d (p , p') = e} .

Then the map S1 (p) ^ S<= (p) : v ^ expp (ev) is a diffeomorphism and , for all q <= M , we have

d (p , q) > e =^ d (S<= (p) , q) = d (p , q) - e .

Proof . By Theorem we have

d (p , expp (v)) = |v| for all v <= TpM with |v| < e and

d (p , p') > e for all p' <= M \ {expp (v) | v <= TpM , |v| < e} .

This shows that S<= (p) = expp (eS1 (p)) and , since e is smaller than the injec - tivity radius , the map

Si (p) ^ S<= (p) : v ^ expp (ev)

is a diffeomorphism .

To prove the second assertion , let q e M such that

r := d (p , q) > e .

Fix a constant 5 > 0 and choose a smooth curve 7 : [0 , 1] ^ M such that

7 (0) = p , 7 (1) = q , l (y) < r + 5 .

Choose to > 0 such that 7 (to) is the last point of the curve on S<= (p) , i . e . Y (to) e S<= (p) , 7 (t) e S<= (p) for to < t < 1 .

Then

d (7 (to) , q) < l (7l[to , i])

= l (7) - l (7|[o , to])

l (7) - e

r + 5 — e .

This shows that d (S<= (p) , q) < r + 5 — e for every 5 > 0 and therefore

d (S<= (p) , q) < r — e .

Moreover ,

d (p' , q) > d (p , q) — d (p , p') = r — e for all p' e S<= (p) . Thus

d (S<= (p) , q) = r — e

and this proves Lemma

Lemma (Curve Shortening Lemma) . Let M C rn be an m - mani - fold , let p e M , and let e be a real number such that

0 < e < inj (p; M) .

Then , for all v , w e TpM , we have

|v| = |w| = e , d (expp (v) , expp (w)) = 2e =^ v + w = 0 .

Two unit tangent vectors .

Proof . We will prove that , for all v , w e TpM , we have

d (expp (5v) , expp (5w))

= |v — w| .

5

Assume this holds and suppose , by contradiction , that there exist two tangent vectors v , w e TpM such that

|v| = |w| = 1 , d (expp (ev) , expp (ew)) = 2e , v + w = 0 .

Then

|v — w| < 2

there exists a constant 0 < 5 < e such that

d (expp (5v) , expp (5w)) < 25 .

Then

d (expp (ev) , expp (ew))

< d (expp (ev) , expp (5v)) + d (expp (5v) , expp (5w)) + d (expp (5w) , expp (ew)) <e — 5 + 25 + e — 5 = 2e

and this contradicts our assumption .

we observe that

d (expp (5v) , expp (5w))

lim p p

Notes

5^0 5

d (expp (5v) , expp (5w)) |expp (5v) — expp (5w) |

= lim i +

5^0 |expp (5v) — expp (5w) | 5

|expp (5v) — expp (5w) 1

= lim J - - L

5^0 5

expp (5v) — p expp (5w) — p

lim

50

= |v — w| .

Here the second equality

Proof of Theorem . By assumption M C rn is a connected submani - fold , and p e M is given such that the exponential map expp : TpM ^ M is defined on the entire tangent space at p . Fix a point q e M \ {p} so that

0 < r := d (p , q) < <xi .

Choose a constant e > 0 smaller than the injectivity radius of M at p and smaller than r . Then , by Lemma , we have

d (S<= (p) , q) = r - e .

Hence there exists a tangent vector v e TpM such that

d (expp (ev) , q) = r — e , |v| = 1 .

Define the curve 7 : [0 , r] ^ M by

Y (t) := expp (tv) for 0 < t < r .

By Lemma , this is a geodesic and it satisfies y (0) = p . We must prove that y (r) = q and L (y) = d (p , q) . Instead we will prove the follow - ing stronger statement .

Claim . For every t e [0 , r] we have

d (y (t) , q) = r — t .

In particular , y (r) = q and L (y) = r = d (p , q) .

Consider the subset

I := {t e [0 , r] | d (Y (t) , q) = r — t} C [0 , r] .

This set is nonempty , because e e I , it is obviously closed , and

t e I =^ [0 , t] C I .

Namely , if t e I and 0 < s < t then

d (Y (s) , q) < d (Y (s) , Y (t)) + d (Y (t) , q) < t — s + r — t = r — s

and

d (Y (s) , q) > d (p , q) — d (p , y (s)) > r — s .

Hence d (Y (s) , q) = r — s and hence s e I .

We prove that I is open (in the relative topology of [0 , r]) . Let t e I be given with t < r . Choose a constant e > 0 smaller than the injectivity radius of M at 7 (t) and smaller than r — t . Then , by Lemma with p replaced by 7 (t) , we have

d (S<= (i (t)) , q) = r — t — e .

Next we choose w e TY (t) M such that

|w| = 1 , d (expY (t) (ew) , q) = r — t — e .

Then

d (Y (t — e) , exp7 (t) (ew)) > d (Y (t — e) , q) — d (exp7W (ew) , q)

Notes

= (r — t + e) — (r — t — e)

= 2e .

The converse inequality is obvious , because both points have distance e to 7 (t)

The proof of the Hopf - Rinow theorem .

Thus we have proved that

d (Y (t — e) , exp7 (t) (ew)) = 2e .

Since

Y (t — e) =exp7 (t) (—e^ (t)) ,

it follows from Lemma that

w = Y (t) .

Hence expY (t) (sw) = Y (t + s) and this implies that

d (Y (t + e) , q) = r — t — e .

Thus t + e e I and , by (4 . 6 . 4) , we have [0 , t + e] e I . Thus we have proved that I is open . In other words , I is a nonempty subset of [0 , r] which is both open and closed , and hence I = [0 , r] .

6.5 GEODESICS IN THE INTRINSIC SETTING…..INTRINSIC DISTANCE

Let M be a connected smooth manifold equipped with a Riemannian metric Then we can define the length of a curve 7 : [0 , 1] ^ M by the formula and it is invariant under reparametrization as in Remark The distance function d : M x M ^ r is then given by the same formula We prove that it still defines a metric on M and that this metric induces the same topology as the smooth STRUCTURE .

Lemma Let M be a connected smooth Riemannian manifold and define the function d : M x M ^ [0 , to) Then d is a metric and induces the same topology as the smooth STRUCTURE .

Proof . The proof has three steps .

Step 1 . Fix a point p0 E M and let $: U ^ Q be a coordinate chart of M onto an open subset Q C rm such that p0 E U . Then there exists an open neighborhood V C U of p0 and constants S , r > 0 such that

5 |$ (p) - 0 (po) | < d (p , po) < 5 - 1 |$ (p) - 0 (po) | (4 . 7 . 1)

for every p E V and d (p , p0) > 5r for every p E M \ V .

Denote the inverse of the coordinate chart $ by ^ := $ - 1 : Q ^ M and define the map g = (gj) ™=i : Q ^ rmxm by gj (x) := (JX (x) , Jj (x)) ^) for x E Q . Then a smooth curve 7 : [0 , 1] ^ U has the length

l (Y) = J Vc (t) Tg (c (t)) c (t) d^ , c (t) := $ (Y (t)) .

Let x0 := $ (p0) E Q and choose r > 0 such that Br (x0) C Q . Then there is a constant 5 E (0 , 1] such that

5 |<=|< V^Tg (x) C < 5 - 1 |<=|

for all x E Br (x0) and <= , g E rm . Define V := $ - 1 (Br (x0)) C U .

Now let p E V and denote x := $ (p) E Br (x0) . Then , for every smooth curve 7 : [0 , 1] ^ V with 7 (0) = p0 and 7 (1) = p , the curve c := $ o 7 takes values in Br (x0) and satisfies c (0) = x0 and c (1) = x

L (y) > 5 f |C (t) | dt > 5 f c (t) dt J 0   J 0

If y : [0 , 1] ^ M is a smooth curve with endpoints y (0) = p0 and y (1) = p whose image is not entirely contained in V then there is a time T e (0 , 1] such that y (t) e V for 0 < t < T and y (T) e dV . Hence c (t) := 0 (y (t)) e Br (x0) for 0 < t < T and |c (T) — x0\ = r . Hence , by the above argument , we have

L (y) > 5r .

This shows that d (p0 , p) > 5r for p e M \ V and d (p0 , p) > 5 \0 (p) — 0 (p0) \ for p e V . If p e V , x := 0 (p) , and c (t) := x0 + t (x — x0) then y := 0 ° c is a smooth curve in V with y (0) = p0 and y (1) = p

This proves Step 1 .

Step 2 . d is a distance function .

Step 1 shows that d (p , p0) > 0 for every p e M \ {p0} and hence d satisfies condition (i) in Lemma . The proofs of (ii) and (iii) remain unchanged in the intrinsic setting and this proves Step 2 .

Step 3 . The topology on M induced by d agrees with the topology induced by the smooth STRUCTURE .

Assume first that W C M is open with respect to the manifold topology and let p0 e W . Let 0 : U ^ Q be a coordinate chart of M onto an open subset Q C rm such that p0 e U , and choose an open neighborhood V C U of p0 and constants 5 , r > 0 as in Step 1 , so that holds for all p e V and d (p , p0) > 5r for every p e M \ V . Then 0 (V n W) is an open subset of Q and so there is an e > 0 such that B^ - ie (0 (p0)) C 0 (V n W) and e < 5r . Let p e M with d (p , p0) < e . Thenp e V , hence \0 (p) — 0 (p0) \ < 5 - 1d (p , p0) and this implies 0 (p) e 0 (Vn W) . Thus Be (p0 , d) C W and so W is open with respect to d .

Conversely , assume that W C M be open with respect to d and choose a coordinate chart 0 : U ^ Q onto an open set Q C rm . We must prove that 0 (U n W) is an open subset of Q . To see this , choose x0 e 0 (U n W) and let p0 := 0 - 1 (x0) e U n W . Now choose V C U and 5 , r > 0 as in Step 1 . Choose e > 0 such that B^ - ie (p0 , d) C W and Be (x0) C 0 (V) . Let x e rn such that \x — x0\ < e . Then x e 0 (V) and therefore p := 0 - 1 (x) e V . This implies d (p , p0) < 5 - 1\0 (p) — 0 (p0) \ = 5 - 1\x — x0\ < 5 - 1e , thus p e W n U , and so x = 0 (p) e 0 (W n U) . Thus 0 (W n U) is an open , and so W is open in the manifold topology of M .

Geodesics and the Levi - Civita Connection

With the covariant derivative understood we can define Geodesics on M as smooth curves 7 : I ^ M that satisfy the equation

V7 = 0 ,

as in Definition . Then all the above results about Geodesics , as well as their proofs , carry over almost verbatim to the intrinsic setting . In particular , geodesics are in local coordinates described by equation and they are the critical points of the energy functional on the space Qp , q of all paths 7 : [0 , 1] ^ M with fixed endpoints 7 (0) = p and 7 (1) = q . Here we use the fact that Lemma extends to the intrinsic setting via the Embedding Theorem So for every vector field X <= Vect (Y) along 7 with X (0) =0 and X (1) =0 there exists a curve of curves r ^ Qp , q : s ^ 7s with 70 = 7 and ds7s |s=0 = X . Then , by the properties of the Levi - Civita connection , we have

The right hand side vanishes for all X if and only if V7 = 0With this understood , we find that , for all p <= M and v <= TpM , there exists a unique geodesic 7 : Ip , v ^ M on a maximal open interval Ip , v C r containing zero that satisfies 7 (0) = p and 7 (0) = v . This gives rise to a smooth exponential map expp : Vp = {v <= TpM | 1 <= Ip , v} ^ M as in which satisfies dexpp (0) = id : TpM ^ TpM as in Corollary . This leads directly to the injectivity radius , the GauB Lemma the local length minimizing property of geodesics in Theorem and the Convex Neighborhood Theorem . Also the proof of the equivalence of metric and geodesic completeness in Theorem and of the Hopf - Rinow Theorem carry over verbatim to the intrinsic setting of general Riemannian manifolds . The only place where some care must be taken is in the proof of the Curve Shortening Lemma as is spelled out in Exercise below .

Examples and Exercises

Exercise . Choose a coordinate chart 0 : U ^ Q with 0 (po) = 0 such that the metric in local coordinates satisfies

gij (0) = 5ij .

Refine the estimate and show that

r d (p , q) = 1

|0 (p) - 0 (q) | .

This is the intrinsic analogue of Lemma Use this to prove that equa - tion continues to hold for all Riemannian manifolds , i . e .

d (expp (5v) expp (^w))

lim = |v — w|

Notes

s^o 5

for p <= M and v , w <= TpM . With this understood , the proof of the Curve Shortening Lemma carries over verbatim to the intrinsic setting .

Exercise . The real projective space rPn inherits a Riemannian met - ric from Sn as it is a quotient of Sn by an isometric involution . Prove that each geodesic in Sn with its standard metric descends to a geodesic in rPn .

Exercise . Let f : S3 ^ S2 be the Hopf fibration defined by

f (z , w) = (jz|2 — |w|2 , 2Re zw , 21m zw^j

Prove that the image of a great circle in S3 is a nonconstant geodesic in S2 if and only if it is orthogonal to the fibers of f , which are also great circles . Here we identify S3 with the unit sphere in c2

Exercise . Prove that a nonconstant geodesic 7 : r ^ S2n + 1 de - scends to a nonconstant geodesic in cPn with the Fubini - Study if and only if 7 (t) T c7 (t) for every t <= r .

Exercise Consider the manifold

Fk (rn) :^D <= rnxfc | DtD = ll}

of orthonormal k - frames in rn , equipped with the Riemannian metric inherited from the standard inner product

{X , Y} := trace (XTY)

on the space of real n x k - matrices .

Check your Progress - 1

Discuss Convexity

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Discuss Minimal Geodesics

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6.6 LET US SUM UP

In this unit we have discussed the definition and example of Convexity , Minimal Geodesics , Completeness and Hopf—Rinow, Geodesics in the Intrinsic Setting… Intrinsic Distance

6.7 KEYWORDS

Convexity ….. A subset of an affine space is called convex iff it contains the line segment joining any two of its points

Minimal Geodesics ….. Any straight line segment in Euclidean space is the shortest curve joining its endpoints

Completeness and Hopf—Rino….. For a Riemannian manifold there are different notions of completeness

Geodesics in the Intrinsic Setting Intrinsic Distance….. Let M be a connected smooth manifold equipped with a Riemannian metric

6.8 QUESTIONS FOR REVIEW

Explain Convexity

Explain Minimal Geodesics

6.9 ANSWERS TO CHECK YOUR PROGRESS

Convexity (answer for Check your Progress - 1 Q)

Minimal Geodesics (answer for Check your Progress - 1 Q)

6.10 REFERENCE

Differential Geometry, Differential Geometry & Application,
Introduction to Defferential Geometry, Basic of Differential Geometry.

UNIT-VII: CURVATURE

STRUCTURE

7.0 Objectives

7.1 Introduction

7.2 Curvature

7. 3 Generalized Theorema Egregium

7. 4 Theorem of Egregium

7.5 Gaussian Curvature

7.6 Let Us Sum Up

7.7 Keywords

7.8 Questions For Review

7.9 Answers To Check Your Progress

7.10 References

7.0 OBJECTIVES

After studying this unit , you should be able to:

- Understand about Curvature
- Generalized Theorem of Egregium
- Theorem of Egregium
- Gaussian Curvature

7.1 INTRODUCTION

Differential geometry arose and developed as a result of and in connection to the mathematical analysis of curves and surfaces Mathematical analysis of curves and surfaces had been developed to answer some of unanswered questions that appeared in calculus like the reasons for relationships between complex shapes and curves , series and analytic functions Curvature , Generalized Theorema Egregium , Theorem of Egregium , Gaussian Curvature

7.2 CURVATURE

This chapter begins by introducing the notion of an isometry . It shows that isometries of embedded manifolds preserve the lengths of curves and can be characterized as diffeomorphisms whose derivatives preserve the inner products . The chapter then moves on to the Riemann curvature tensor . The next section is devoted to the generalized GauB Theorema Egregium which asserts that isometries preserve geodesics , the covariant derivative , and the Riemann curvature tensor . That section also shows that isometries form finite dimensional Lie groups . The final section discusses the Riemann curvature tensor in local coordinates and shows how all the definitions and results of the present chapter carry over to the intrinsic setting of Riemannian manifolds

Isometries

Let M and M' be connected submanifolds of rn . An isometry is an isomor - phism of the intrinsic geometries of M and M' . Recall the definition of the intrinsic distance function

d : M x M ^ [0 , x)

in by

for p , q <= M . Let d' denote the intrinisic distance function on M' .

Theorem (Isometries) . Let $: M ^ M' be a bijective map . Then the following are equivalent .

$ intertwines the distance functions on M and M' , i . e .

d' ($ (p) , $ (q)) = d (p , q)

for all p , q e M .

$ is a diffeomorphism and

df (p) : TPM ^ T^ (p) M'

is an orthogonal isomorphism for every p e M .

$ is a diffeomorphism and

L (f o Y) = l (y) for every smooth curve y : [a , b] ^ M .

The bijection $ is called an isometry if it satisfies these equivalent condi - tions . In the case M = M1 the isometries $: M ^ M form a group denoted by I (M) and called the isometry group of M .

Lemma . For every p e M there exists a constant e > 0 such that , for all v , w e TpM with 0 < |w| < |v| < e , we have

|w|

d (expp (w) , expp (v)) = |v| — |w| =^ w = - j - j - v . (5 . 1 . 1)

Remark . It follows from the triangle inequality and Theorem that

d (expp (v) , expp (w)) > d (expp (v) , p) — d (expp (w) , p) = |v| — |w|

whenever 0 < |w| < |v| < inj (p) . Lemma asserts that equality can only hold when w is a positive multiple of v or , to put it differently , that the distance between expp (v) and expp (w) must be strictly bigger that |v| — |w| whenever w is not a positive multiple of v .

Proof of Lemma . As in Corollary we denote

Be (p) := {v e TpM | |v| < e} ,

Ue (p) := {q e M | d (p , q) < e} .

By Theorem and the definition of the injectivity radius , the exponential map at p is a diffeomorphism expp : Be (p) ^ Ue (p) for e < inj (p) . Choose 0 < r < inj (p) . Then the closure of Ur (p) is a compact subset of M . Hence there is a constant e > 0 such that e < r and e < inj (p') for every p' e Ur (p) . Since e < r we have

e < inj (p') V p' e Ue (p) .

Thus expp / : Be (p') ^ Ue (p') is a diffeomorphism for every p' e Ue (p) . Define pi := expp (w) and p2 := expp (v) . Then , by assumption , we have d (pi , p2) = |v| — |w| < e . Since p1 e Ue (p) it follows from our choice of e that e < inj (p1) . Hence there is a unique tangent vector v1 e Tpi M such that

|v11 = d (pi , p2) = |v| — |w| , expp1 (vi) = p2 .

Following first the shortest geodesic from p to p1 and then the shortest geodesic from p1 to p2 we obtain (after suitable reparametrization) a smooth Y : [0 , 2] ^ M such that

Y (0) = p , Y (1) = p1 , y (2) = p2 ,

and

l (Y|[0 , 1]) = d (p , p1) = |w| , l (Y|[1 , 2]) = d (p1 , p2) = |v| — |w| .

Thus L (y) = |v| = d (p , p2) . Hence , by Theorem there is a smooth function P : [0 , 2] ^ [0 , 1] satisfying

P (0) = 0 , P (2) = 1 , / 3 (t) > 0 , Y (f) =expp (^ (t) v)

for every t e [0 , 2] . This implies

expp (w) = p1 = y (1) = expp (P (1) v) , 0 < P (1) < 1 .

Since w and P (1) v are both elements of Be (p) and expp is injective on Be (p) , this implies w = P (1) v . Since P (1) > 0 we have P (1) = |w| / |v| .

Proof of Theorem follows from the definition of the length of a curve . Namely

r b

L (f oy) =b

\df (Y (t)) 7 (t) \ dt

b

= / \7 (t) \ dt

J a

= l (y) .

In the third equation we have used (ii) . That (iii) implies (i) follows immediately from the definition of the intrinsic distance functions d and d! .

We prove that (i) implies (ii) . Fix a point p e M and choose e > 0 so small that e < inj (p) and that the assertion of Lemma holds for the point p' := f (p) e M' . Then there is a unique homeomorphism

: B (p) ^ B<= (f (p))

and f : Us (p) ^ Us (f (p)) is a homeomorphism by (i) . Hence Tp : Bs (p) ^ Bs (f (p)) is a homeomorphism .

Claim 1 . The map Tp satisfies the following equations for every v e Bs (p) and every t e [0 , 1]:

exp^ (p) ($p (v)) = f (expp (v))

\Tp (v) \ = d' (f (p) , exp^p) (Tp (v)))

= d' (f (p) , f (expp (v)))

= d (p , expp (v))

= |v| .

Here the second equation follows and the third equation from (i) . Equation holds for t = 0 because Tp (0) = 0 and for t = 1 it is a tautology . Hence assume 0 < t < 1 . Then

d / (exp^ (p) ($P (tv)) exP^ (P} ($p (v))) = d' (^ (exPp (tv)) , ^ (exPp (v)))

= d (expp (tv) , expp (v)) = |v| — |tv|

= |$p (v) | — |$p (tv) | .

Here the first equation follows from the second equation from (i) , the third equation from Theorem and the fact that |v| < inj (p) , and the last equation follows Since 0 < |Tp (tv) | < |Tp (v) | < e we can apply Lemma and obtain

*p (tv) = Ivtf *p (v) =t<Mv) .

This proves Claim 1 .

By Claim 1 , Tp extends to a bijective map Tp : TpM ^ T^ (p) M / via

where 5 > 0 is chosen so small that 5 |v| < e . The right hand side of this equation is independent of the choice of 5 . Hence the extension is well defined . It is bijective because the original map Tp is a bijection from Be (p) to Be (0 (p)) for all v E TpM and all t > 0 .

Claim 2 . The extended map Tp : TpM ^ T^ (p) M / is linear and preserves the inner product .

d (expp (tv) , expp (tw))

|v — w| = lim

t^ 0

t

d / (^ (expp (tv)) , ^ (expp (tw))) t

d' (exp^P) ($p (tv))) , exp^ (p) ($p (tw))))

t

d / (exp^p) (t$p (v))) , exp^ (p) (t$p (w))))

t

|$p (v) — $p (w) | .

2{v , w) = |v|2 + |w|2 - |v - w|2

= |Tp (v) |2 + |Tp (w) |2 - |Tp (v) - Tp (w) |2 = 2{Tp (v) , Tp (w))
.

Thus Tp preserves the inner product . Hence , for all v\ , v2 , w <= TpM , we have

(Tp (v i + v2) , Tp (w)) = {vi + V2 , w)

= (vi , w) + (v2 , w)

= (Tp (vi) , Tp (w)) + (Tp (v2) , Tp (w)) = (Tp (vi) + Tp (v2) , Tp (w)) .

Since Tp is surjective , this implies

Tp (vi + v2) = Tp (vi) + Tp (v2)

for all vi , v2 <= TpM . With vi = v and v2 = —v we obtain

Tp (—v) = —Tp (v)

for every v <= TpM and by this gives

Tp (tv) = tTp (v) for all v <= TpM and t <= r . This proves Claim 2 .

Claim 3 . 0 is smooth and d0 (p) = Tp .

By we have

0 = exp^ (p) °Tp ° exp - i : U^ (p) ^ U<= (0 (P)) .

Since Tp is linear , this shows that the restriction of 0 to the open set U<= (p) is smooth . Moreover , for every v <= TpM we haveexp^ (p) (tTp (v)) = Tp (v) .

Exercise . Prove that every isometry 0 : rn ^ rn is an affine map

0 (p) = Ap + b

where A <= O (n) and b <= rn . Thus 0 is a composition of translation and rotation . Hint: Let ei , . . . , en be the standard basis of rm . Prove that any two vectors v , w <= rn that satisfy

|v| = |w|

and

|v — ej| = |w — ej| for i = 1 , . . . , n

must be equal .

Remark . If 0 : rn ^ rn is an isometry of the ambient Euclidean space with 0 (M) = M' then certainly 0 := 0|m is an isometry from M onto M' . On the other hand , if M is a plane manifold

M = { (0 , y , z) <= r3 | 0 < y < n / 2}

and M' is the cylindrical manifold

M' = { (x , y , z) <= r3 | x2 + y2 = 1 , x > 0 , y > 0}

Then the map 0 : M ^ M' defined by

0 (0 , y , z) := (cos (y) , sin (y) , z)

is an isometry which is not of the form 0 = 0|m . Indeed , an isometry of the form 0 = 0|M necessarily preserves the second fundamental form (as well as the first) in the sense that

d^ (p) hp (v , w) = h^ (p) (d0 (p) v , d0 (p) w)

for v , w <= TpM but in the example h vanishes identically while h' does not . We may thus distinguish two fundamental question:

I . Given M and M' when are they extrinsically isomorphic , i . e . when is there an ambient isometry 0 : rn ^ rn with 0 (M) = M'?

II . Given M and M' when are they intrinsically isomorphic , i . e . when is there an isometry 0 : M ^ M' from M onto M'?

As we have noted , both the first and second fundamental forms are preserved by extrinsic isomorphisms while only the first fundamental form need be preserved by an intrinsic isomorphism (i . e . an isometry) .

A question which occurred to GauB (who worked for a while as a cartographer) is this: Can one draw a perfectly accurate map of a portion of the earth? (i . e . a map for which the distance between points on the map is proportional to the distance between the corresponding points on the surface of the earth) . We can now pose this question as follows: Is there an isometry from an open subset of a sphere to an open subset of a plane? GauB answered this question negatively by associating an invariant , the GauBian curvature

K : M ^ r ,

to a surface M c r3 . According to his Theorema Egregium

K' o 0 = K

for an isometry 0 : M ^ M' . The sphere has positive curvature; the plane has zero curvature; hence the perfectly accurate map does not exist . Our aim is to explain these ideas .

We shall need a concept slightly more general than that of "isometry" .

Definition 5 . 1 . 6 . A smooth map 0 : M ^ M' is called a local isometry

if its derivative

d0 (p) : TpM ^ T^ (p) M' is an orthogonal linear isomorphism for every p e M .

Remark . Let M c rn and M' c rm' be manifolds and 0 : M ^ M' be a map . The following are equivalent . 0 is a local isometry .

For every p e M there are open neighborhoods U C M and U' C M' such that the restriction of 0 to U is an isometry from U onto U' .

Example . The map

r ^ S1 : d ^ eie is a local isometry but not an isometry .

Exercise Let M C rn be a compact connected 1 - manifold . Prove that M is diffeomorphic to the circle S1 . Define the length of a compact connected Riemannian 1 - manifold . Prove that two compact connected 1 - manifolds M , M' C rn are isometric if and only if they have the same length . Hint: Let 7 : r ^ M be a geodesic with | / y (t) | = 1 . Show that Y is not injective; otherwise construct an open cover of M without finite subcover . If to < t1 with y (to) = Y (t1) show that Y (to) = 7 (H) ; otherwise show that y (t0 + 1) = y (t1 — t) for all t and find a contradiction .

We close this section with a result which asserts that two local isometries that have the same value and the same derivative at a single point must agree everywhere , provided that the domain is connected .

Notes

Lemma . Let M C rn and M' C rn' be smooth m - manifolds and assume that M is connected . Let

0 : M ^ M' , ^ : M ^ M'

be local isometries and let po e M such that

0 (po) = ^ (po) =: p'o , d0 (Po) = d^ (po) : TpoM ^ Tp / QM' .

Then 0 (p) = ^ (p) for every p e M .

Proof . Define the set

Mo := {p e M | 0 (p) = ^ (p) , d<fi (p) = d^ (p) } .

This set is obviously closed . We prove that Mo is open . Let p e Mo and choose U C M and U' C M' as in Remark . Denote

:= d0 (p) = d^ (p) : TpM ^ TpM' , p' := 0 (p) = ^ (p)

there exists a constant e > 0 such that U<= (p) C U and U<= (p') C U' and

q e U<= (p) =^ 0 (q) =expp , oTp o exp - 1 (q) = ^ (q) .

Hence U<= (p) C Mo . Thus Mo is open , closed , and nonempty . Since M is connected it follows that Mo = M Riemann Curvature Tensor Definition and Gaufi—Codazzi

Let M c rn be a smooth manifold and 7 : r2 ^ M be a smooth map . Denote by (s , t) the coordinates on r2 . Let Z e Vect (y) be a smooth vector field along 7 , i . e . Z : r2 ^ rn is a smooth map such that Z (s , t) e TY (s , t) M for all s and t . The covariant partial derivatives of Z with respect to the variables s and t are defined by

dZ dZ

V*Z = n (Y) Ids - VZ := n (7) w .

In particular ds7 = d^ / ds and dtY = 87 / dt are vector fields along 7 and we have

VsdtY - VtdsY = 0

as both terms on the left are equal to n (7) dsdt7 . Thus ordinary partial differentiation and covariant partial differentiation commute . The analogous formula (which results on replacing d by V and 7 by Z) is in general false . Instead we have the following .

Definition . The Riemann curvature tensor assigns to eachp e M the bilinear map

Rp : TpM x TpM ^ L (TpM , TpM)

characterized by the equation

Rp (u , v) w = (VsVtZ - VtVsZ) (0 , 0)

for u , v , w e TpM where 7 : r2 ^ M is a smooth map and Z e Vect (q) is a smooth vector field along 7 such that

Y (0 , 0) = p , dsY (0 , 0) = u , dtY (0 , 0) = v , Z (0 , 0) = w .

We must prove that R is well defined , i . e . that the right hand side of equation is independent of the choice of 7 and Z . This follows from the GauB - Codazzi formula which we prove next . Recall that the second fun - damental form can be viewed as a linear map hp : TpM ^ L (TpM , TpM^) and that , for u e TpM , the linear map hp (u) e L (TpM , TpMx) and its dual hp (u) * e L (TpM± , TpM) are given by

hp (u) v = (dn (p) u) v , hp (u) *w = (dn (p) u) w

for v e TpM and w e TpMx .

Theorem The Riemann curvature tensor is well defined and given by the GauB—Codazzi formula

Rp (u , v) = hp (u) *hp (v) — hp (v) *hp (u)

for u , v e TpM .

Proof . Let u , v , w e TpM and choose a smooth map Y : r2 ^ M and a smooth vector field Z along 7 . Then , by the GauB Weingarten , we have

VtZ = dtZ — h7 (dti) z = dtZ — (dII (7) dt7) Z = dtZ — (dt (n o 7)) Z .

Notes

Hence

OsVtZ = dsdtZ — ds ((dt (n o 7)) z)

= dsdtZ — (dsdt (n o 7)) Z — (dt (n o 7)) dsZ = dsdtZ — (dsdt (n o 7)) Z — (dn (7) dt7) (VsZ + h7 (ds7) Z) = dsdtZ — (dsdt (U o y)) Z — h7 (dtY) VsZ — h7 (dt7) *h7 (dsY) Z .

Interchanging s and t and taking the difference we obtain

ds VtZ — dtVsZ = h7 (dsY) *h7 (dtY) Z — h7 (dtY) * h7 (dsY) Z + h7 (dsY) Vt Z — h7 (dtY) VsZ .

Here the first two terms on the right are tangent to M and the last two terms on the right are orthogonal to TYM . Hence

Vs VtZ — VtVsZ = n (Y) (dsVtZ — dtVsZ)

= h7 (dsY) *h7 (dtY) Z — h7 (dtY) *h7 (dsY) Z .

Evaluating the right hand side at s = t = 0 we find that

(VsVtZ — VtVsZ) (0 , 0) = hp (u) *hp (v) w — hp (v) *hp (u) w .

This proves the GauB - Codazzi equation and shows that the left hand side is independent of the choice of Y and Z .

Covariant Derivative of a Global Vector Field

So far we have only defined the covariant derivatives of vector fields along curves . The same method can be applied to global vector fields . This leads to the following definition .

Definition (Covariant derivative) . Let M C rn be an m - dimen - sional submanifold and X be a vector field on M . Fix a point p e M and a tangent vector v e TpM . The covariant derivative of X at p in the

direction v is the tangent vector

VX (p) := n (p) dX (p) v e TPM , where n (p) e rnxn denotes the orthogonal projection onto TpM .

Remark . If 7 : I ^ M is a smooth curve on an interval I C r and X e Vect (M) is a smooth vector field on M then X o 7 is a smooth vector field along 7 . The covariant derivative of X o 7 is related to the covariant derivative of X by the formula

V (X o 7) (t) = Vy (t) X (7 (t)) .

Remark (GauB—Weingarten formula) . Differentiating the equation X = nX (understood as a function from M to rn) and using the notation dvX (p) := dX (p) v for the derivative of X at p in the direction v we obtain the GauB—Weingarten formula for global vector fields:

dv X (p) = Vv X (p) + hp (v) X (p) .

Remark 5 . 2 . 6 (Levi - Civita connection) . Differentiating a vector field Y on M in the direction of another vector field X we obtain a vector field VXY e Vect (M) defined by

(VxY) (p) := Vx (p) Y (p)

for p e M . This gives rise to a family of linear operators

Vx : Vect (M) ^ Vect (M) ,

one for every vector field X e Vect (M) , and the assignment

Vect (M) ^ L (Vect (M) , Vect (M)) : X ^ Vx

is itself a linear operator . This operator is called the Levi - Civita connection on the tangent bundle TM . It satisfies the conditions

V / x (Y) = f Vx Y , (5 . 2 . 6)

Vx (fY) = fVxY + (Lxf) Y (5 . 2 . 7)

Lx (Y , Z) = (VxY , Z) + (Y , VxZ) , (5 . 2 . 8)

Vy X - Vx Y = [X , Y] , (5 . 2 . 9)

for all X , Y , Z e Vect (M) and f e F (M) , where LXf = df o X and [X , Y] e Vect (M) denotes the Lie bracket of the vector fields X and Y . The next lemma asserts that the Levi - Civita connection is uniquely deter - mined .

Lemma (Uniqueness Lemma) . There is a unique linear operator

Vect (M) ^ L (Vect (M) , Vect (M)) : X ^ Vx satisfying equations for all X , Y , Z e Vect (M) .

Proof . Existence follows from the properties of the Levi - Civita connection . We prove uniqueness . Let X ^ DX be any linear operator from Vect (M) to L (Vect (M) , Vect (M)) that satisfies . Then we have

Lx (Y , Z) = (DxY , Z) + (Y , DxZ) ,

Ly (X , Z) = (DyX , Z) + (X , DyZ) ,

- Lz (X , Y) = - (DzX , Y) - (X , Dz Y) .

Adding these three equations we find

Lx (Y , Z) + Ly (Z , X) - Lz (X , Y)

= 2 (Dx Y , Z) + (DyX - DxY , Z)

+ (X , DyZ - DzY) + (Y , DxZ - DzX)

= 2 (DxY , Z) + ([X , Y] , Z) + (X , [Z , Y]) + (Y , [Z , X]) .

The same equation holds for the Levi - Civita connection and hence

(DxY , Z) = (VxY , Z) .

This implies DXY = VXY for all X , Y e Vect (M) . □

Remark (The Levi - Civita connection in local coordinates) .

Let 0 : U ^ 0 be a coordinate chart on an open set U C M with values in an open set 0 C rm . In such a coordinate chart a vector field X e Vect (M) is represented by a smooth map

<= = (A , . . . , <= m) :0 ^ rm

defined by

<= (0 (p)) = d<ft (p) X (p)

for p e U . If Y e Vect (M) is represented by n then VXY is represented by the function

m o k m

(Vfn) k := E dXi + E 4«V .

i= 1 i , j= 1

Here the rkj : 0 ^ r are the Christoffel symbols defined by

rk := kt 1 (d9a . dg<=j %j\ (52ii)

ij :=^ - g HdXj + up - d¥ , i=i v 7

where gij is the metric tensor and gij is the inverse matrix so that

E gij gjk = Jk

j

This formula can be used to prove the existence statement in Lemma and hence define the Levi - Civita connection in the intrinsic setting .

Exercise In the proof of Lemma we did not actually use that the operator

Dx : Vect (M) ^ Vect (M) is linear nor that the operator X ^ Dx is linear . Prove directly that if a map

Dx : L (M) ^ L (M)

satisfies for all Y , Z e Vect (M) then DX is linear . Prove that every map Vect (M) ^ L (Vect (M) , Vect (M)) : X ^ Dx that satisfies is linear .

A Global Formula

Lemma For X , Y , Z <= Vect (M) we have

R (X , Y) Z = XxXyZ — XyXxZ + X[x , y]Z .

Proof . Fix a point p <= M . Then the right hand side of equation at p remains unchanged if we multiply each of the vector fields X , Y , Z by a smooth function f : M ^ [0 , 1] that is equal to one near p . Choosing f with compact support we may therefore assume that the vector fields X and Y are complete . Let <fs denote the flow of X and f) t the flow of Y . Define the map y : r2 ^ M by

Y (s , t) := <ps o f) t (p) , s , t <= r .

Then

dsY = X (Y^ dtY = (0*Y) (Y) .

Hence , by Remark , we have

Xs (Z o y) = (XxZ) (y) , Xt (Z o y) = (X^ - yZ) (y) .

This implies

XsXt (Z o y) = (Xds7X^syZ) (y) + (Xds^jyZ) (y) .

Since

d

^s Y = [X , Y]

s

* Y s=0

and dsY = X (y) we obtain

XsXt (Z o y) (0 , 0) = XxXyZ (p) + X[x , y]Z (p) , XtXs (Z o y) (0 , 0) = XyXxZ (p) .

Hence

RP (X (p) , Y (p)) Z (p) = (XsXt (Z o y) — XtXs (Z o y^ (0 , 0)

= Xx Xy Z (p) — Xy Xx Z (p) + X[x , y] Z (p) .

Remark Equation can be written succinctly as

[Vx , W] + Vx , y] = R (X , Y) . This can be contrasted with the equation

[Lx , Ly] + C[x , y] =0 for the operator LX on the space of real valued functions on M .

Remark Equation can be used to define the Riemann curvature tensor . To do this one must again prove that the right hand side of equation at p depends only on the values X (p) , Y (p) , Z (p) of the vector fields X , Y , Z at the point p . For this it suffices to prove that the map

Vect (M) x Vect (M) x Vect (M) ^ Vect (M) : (X , Y , Z) ^ R (X , Y) Z is linear over the Ring F (M) of smooth real valued functions on M , i . e .

R (fX , Y) Z = R (X , fY) Z = R (X , Y) fZ = fR (X , Y) Z

for X , Y , Z e Vect (M) and f e F (M) . The formula follows from the equations , and [X , fY] = f [X , Y] — (LXf) Y . It follows from that the right hand side of at p depends only on the vectors X (p) , Y (p) , Z (p) . The proof requires two steps . One first shows that if X vanishes near p then the right hand side of vanishes at p (and similarly for Y and Z) . Just multiply X by a smooth function equal to zero at p and equal to one on the support of X; then fX = X and hence the vector field R (X , Y) Z = R (fX , Y) Z = fR (X , Y) Z vanishes at p . Second , we choose a local frame Ei , . . . , Em e Vect (M) , i . e . vector fields that form a basis of TpM for each p in some open set U C M . Then we may write

m m m

X = <= fEi , Y = <= Ej , Z = <= Zk Ek

i=1 j=1 k=1

in U . Using the first step and the F (M) - multilinearity we obtain

m

R (X , Y) Z = <= einjZkR (Ei , Ej) Ek

i , j , k=1

in U . If X' (p) = X (p) then Zi (p) = Z;i (p) so if X (p) = X' (p) , Y (p) = Y' (p) , Z (p) = Z' (p) then (R (X , Y) Z) (p) = (R (X' , Y') Z') (p) as required .

Symmetries

Notes

Theorem . The Riemann curvature tensor satisfies

R (Y , X) = - R (X , Y) = R (X , Y) * , R (X , Y) Z + R (Y , Z) X + R (Z , X) Y = 0 , {R (X , Y) Z , W) = {R (Z , W) X , Y) ,

for X , Y , Z , W <= Vect (M) . Equation is the first Bianchi identity .

Proof . The first equation in is obvious from the definition and the second follows from the GauB - Codazzi formula . Alternatively , choose a smooth map 7 : r2 ^ M and two vector fields Z , W along 7 . Then

0 = Osdt{Z , W) - Otds{Z , W)

= ds{VtZ , w) + ds{Z , Vt w) - dt{VsZ , w) - ot{Z , VsW)

= {VsVtZ , W) + {Z , VsVtW) - {VtVsZ , W) - {Z , VtVsW)

= {R (dsY , dtY) Z , W) - {Z , R (dsY , dtY) W) .

This proof has the advantage that it carries over to the intrinsic setting . We

prove the first Bianchi identity:

R (X , Y) Z + R (Y , Z) X + R (Z , X) Y

= VxVyZ - VyVxZ + VfoY]Z + VyVzX - VzVyX + V[y , z]X

+ Vz Vx Y - Vx Vz Y + V[z , x]Y = Vy , z]X - Vx[Y , Z] + Vz , x] Y - Vy [Z , X] + Vx , y]Z - Vz[X , Y] = [X , [Y , Z]] + [Y , [Z , X]] + [Z , [X , Y]] .

The last term vanishes by the Jacobi identity . We prove by combining the first Bianchi identity

{R (X , Y) Z , W) - {R (Z , W) X , Y)

= - {R (Y , Z) X , W) - {R (Z , X) Y , W) - {R (Z , W) X , Y) = {R (Y , Z) W , X) + {R (Z , X) W , Y) + {R (W , Z) X , Y) = {R (Y , Z) W , X) - {R (X , W) Z , Y)

= {R (Y , Z) W , X) - {R (W , X) Y , Z) .

Note that the first line is related to the last by a cyclic permutation . Repeating this argument we find

{R (Y , Z) W , X) - {R (W , X) Y , Z) = {R (Z , W) X , Y) - {R (X , Y) Z , W) .

Remark . We may think of a vector field X on M as a section of the tangent bundle . This is reflected in the alternative notation

Q° (M , TM) := Vect (M) .

A 1 - form on M with values in the tangent bundle is a collection of linear maps A (p) : TpM ^ TpM , one for every p <= M , which is smooth in the sense that for every smooth vector field X and M the assignment p ^ A (p) X (p) defines again a smooth vector field on M . We denote by

Q1 (M , TM)

the space of smooth 1 - forms on M with values in TM . The covariant deriva - tive of a vector field Y is such a 1 - form with values in the tangent bundle which assigns to every p <= M the linear map TpM ^ TpM : v ^ VvY (p) . Thus we can think of the covariant derivative as a linear operator

V : Q° (M , TM) ^ Q1 (M , TM) .

The equation asserts that the operators X ^ VX indeed determine a linear operator from Q° (M , TM) to Q1 (M , TM) . Equation asserts that this linear operator V is a connection on the tangent bundle of M . Equation asserts that V is a Riemannian connection and equation asserts that V is torsion - free . Thus Lemma can be restated as asserting that the Levi - Civita connection is the unique torsion - free Riemannian connection on the tangent bundle .

Exercise . Extend the notion of a connection to a general vector bun - dle E , both as a collection of linear operators VX : Q° (M , E) ^ Q° (M , E) , one for every vector field X <= Vect (M) , and as a linear operator

V : Q° (M , E) ^ Q1 (M , E)

satisfying the analogue of equation . Interpret this equation as a Leibniz rule for the product of a function on M with a section of E . Show that Vp is a connection on TMp . Extend the notion of curvature to connections on general vector bundles .

Exercise . Show that the field which assigns to each p <= M the multi - linear map Ap : TpM x TpM ^ L (TpMp , TpMp) characterized by

Ap (dsY , dty) Y = V^V^Y - V^V^Y for y : r2 ^ M and Y <= Vectp (Y) satisfies the equation Ap (u , v) = hp (u) hp (v) * — hp (v) hp (u) * for p <= M and u , v <= TpM .

Examples and Exercises

Example . Let G C O (n) be a Lie subgroup , i . e . a subgroup that is also a submanifold . Consider the Riemannian metric on G induced by the inner product

(v , w) := trace (vTw)

on the ambient space g ((n , r) = rnxn . Let g := Lie (G) = TjG be the Lie algebra of G . Then the Riemann curvature tensor on G can be expressed in terms of the Lie bracket (see item (d) below) .

The maps g ^ ag , g ^ ga , g ^ g - 1 are isometries of G for every a e G .

A smooth map 7 : r ^ G is a geodesic if and only if there exist matrices g e G and <= e g such that

Y (t) = g exp (t<=) .

For G = O (n) we have seen this in Example and the proof in the general case is similar . Hence the exponential map exp : g ^ G defined by the exponential matrix agrees with the time - 1 - map of the geodesic flow .

Let y : r ^ G be a smooth curve and X e Vect (Y) be a smooth vector field along y . Then the covariant derivative of X is given by

d 1

Y (t) - 1VX (t) = —Y (t) - 1x (t) + 2 [Y (t) - 1^ (t) , Y (t) - 1x (t)] .

(Exercise: Prove equation . Hint: Since g C o (n) we have the identity trace ((<=n + n<=) Z) = 0 for all <= , n , Z e g .)

The Riemann curvature tensor on G is given by

g - 1Rg (u , v) w = - 4[[g - 1u , g - 1v] , g - 1w] .

Note that the first Bianchi identity is equivalent to the Jacobi identity .

Exercise . Prove that every Lie subgroup of O (n) is a closed subset and hence is compact . Show that the inner product on the Lie algebra g = Lie (G) = TjG of a Lie subgroup G C O (n) is invariant under conjugation:

(<= , n) = {g<=g - 1 , gng - 1)

for all g e G and all <= , n e g . Show that

{[<= , n] , C) = {^ [n , Z])

for all <= , n , Z e g .

Example . Let G C GL (n , r) be any Lie subgroup , not necessarily contained in O (n) , and let

g := Lie (G) = TtG

be its Lie algebra . Fix any inner product on the Lie algebra g (not necessarily invariant under conjugation) and consider the Riemannian metric on G defined by

(v , w) g := (vg - 1 , wg - 1)

for v , w <= TgG . This metric is called right invariant .

Define the linear map A : g ^ End (g) by

W ( ) n , Z) = 2 (({ , in . Cl) - (n , [C . €l) - « . I€ . n]))

for Z , n , Z <= g . Then A is the unique linear map that satisfies

A (Z) + A (Z) * = 0 , A (n) Z + A (Z) n = [Z , n]

for all Z , n <= g . Here A (Z) * denotes the adjoint operator with respect to the given inner product on g . Note that A (Z) n = — 1lZ , n] whenever the inner product on g is invariant under conjugation .

Notes

Let y : r ^ G be a smooth curve and X <= Vect (y) be a smooth vector field along y . Then the covariant derivative of X is given by

VX = (dt (Xy - 1) + A (^7 - 1) Xy - ^ y .

Hence a smooth curve y : r ^ G is a geodesic if and only if it satisfies the equation

dt (TY - 1) + a (TY - 1) TY - 1 = 0 .

The Riemann curvature tensor on G is given by

(Rg (u , v) w) g - 1 = (A ([ug - 1 , vg - 1]) + [A (ug - 1) , A (vg - 1)]) wg - 1 for g <= G and u , v , w <= TgG .

7.3 GENERALIZED THEOREM OF EGREGIUM

We will now show that Geodesics , covariant differentiation , parallel trans - port , and the Riemann curvature tensor are all intrinsic , i . e . they are in - tertwined by isometries . In the extrinsic setting these results are somewhat surprising since these objects are all defined using the second fundamental form , whereas isometries need not preserve the second fundamental form in any sense but only the first fundamental form .

Below we shall give a formula expressing the GauBian curvature of a surface M2 in r3 in terms of the Riemann curvature tensor and the first fundamental form . It follows that the GauBian curvature is also intrinsic . This fact was called by GauB the "Theorema Egregium" which explains the title of this section .

Pushforward

We assume throughout this section that M c rn and M' c rn' are smooth submanifolds of the same dimension m . As in §5 . 1 we denote objects on M' by the same letters as objects in M with primes affixed . In particular , g' denotes the first fundamental form on M' and R' denotes the Riemann curvature tensor on M' .

Let 0 : M ^ M' be a diffeomorphism . Using 0 we can move objects on M to M' . For example the pushforward of a smooth curve 7 : I ^ M is the curve

0*Y := 0 o 7 : I ^ M' , the pushforward of a smooth function f : M ^ r is the function

0*f := f o 0 - 1 : M' ^ r ,

the pushforward of a vector field X e Vect (Y) along a curve 7 : I ^ M is the vector field 0*X e Vect (0*Y) defined by

(0*X) (t) := d0 (7 (t)) X (t)

for t e I , and the pushforward of a global vector field X e Vect (M) is the vector field 0*X e Vect (M') defined by

(0*X) (0 (p)) := d0 (p) X (p)

for p e M . Recall that the first fundamental form on M is the Riemannian metric g defined as the restriction of the Euclidean inner product on the ambient space to each tangent space of M . It assigns to each p e M the bilinear map gp e TpM x TpM ^ r given by

gp (u , v) = {u , v} , u , v e TpM .

Its pushforward is the Riemannian metric which assigns to each p' e M' the inner product (0*g) p : TpM' x Tp M' ^ r defined by

(0*g) ^ (p) (d0 (p) u , d0 (p) v) := gp (u , v)

for p := 0 - 1 (p') e M and u , v e TpM . The pushforward of the Riemann curvature tensor is the tensor which assigns to each p' e M' the bilinear map (0*R) p : Tp / M' x TpM' ^ L (Tp / M' , TpM') , defined by

(<p*R) pp) (d<p (p) u , d0 (p) v) := d0 (p) Rp (u , v) d0 (p) - 1

for p := 0 - 1 (p') e M and u , v e TpM .

7.4 THEOREM OF EGREGIUM

Notes

Theorem (Theorem of Egregium) . The first fundamental form , covariant differentiation , Geodesics , parallel transport , and the Riemann curvature tensor are intrinsic . This means that for every isometry 0 : M ^ M' the following holds .

0*g = g' .

If X e Vect (y) is a vector field along a smooth curve 7 : I ^ M then

V' (0* X) = 0*VX

and if X , Y e Vect (M) are global vector fields then

VU 0*Y = 0* (Vx Y) .

If Y : I ^ M is a geodesic then 0 o 7 : I ^ M' is a geodesic .

If Y : I ^ M is a smooth curve then for all s , t e I:

^o7 (t , s) d0 (Y (s)) = d0 (Y (t)) ^7 (t , s) .

(v) 0*R = R' .

Proof . Assertion (i) is simply a restatement of Theorem . To prove (ii) we choose a local smooth parametrization / : Q ^ U of an open set U C M , defined on an open set Q C rm , so that / - 1 : U ^ Q is a coordinate chart . Suppose without loss of generality that y (t) e U for all t e I and define c : I ^ Q and <= : I ^ rm by

m d /

Y (t) = / (c (t)) X (t) = E <=i (t) dx (c (t)) -

i= 1

VX (t) = E I <= (t) + E rk (c (t)) ci (t) <=i (t)) JY (c (t)) ,

k=1 \ i , j=1 /

where the Christoffel symbols rj : Q ^ r are defined by

n (/) = E rk ^ .

dxidxj ^ iJ dxk

k=1

Now consider the same formula for f *X using the parametrization

/ ' := f o / : Q ^ U' := f (U) C M' .

The Christoffel symbols r'k : Q ^ r associated to this parametrization of U' are defined by the same formula as the rk - with / replaced by / ' . But the metric tensor for / agrees with the metric tensor for / ':

/ d / d / \ / d / ' d / ' \ gij \ dxi , dxj / \ dxi , dxj / .

Hence it follows from Lemma 3 . 6 . 5 that r'k = rk - for all i , j , k . This implies that the covariant derivative of f *X is given by

m I m \ d / '

v' (f . x) = Euk + E T <c) cr) (c)

k=1 \ i , j=1 /

mm

= df (/ (c)) E (<=k + E 4 (c) YJ) X (c)

yvw - s | dxk

k=1 \ i , j=1

= f*VX .

Here is a second proof of (ii) . For every vector field X e Vect (M) we define the operator DX : Vect (M) ^ Vect (M) by

DxY := 0* (%x0*Y) .

Then , for all X , Y e Vect (M) , we have

DyX - DxY = 0* (V^y0*X - V^x0*Y) = 0*[0*X , 0*Y] = [X , Y] .

Moreover , it follows from (i) that

0*Lx (Y , Z) = L^x (0*Y , 0*Z)

Notes

= (V^x 0*Y , 0*Z) + (0*Y , V^x 0*Z) = (0 (Dx Y , 0*Z) + (0*Y , 0* Dx Z)

= 0* ((DxY , Z) + (Y , DxZ)) .

and hence Lx (Y , Z) = (DxY , Z) + (Y , DxZ) for all X , Y , Z e Vect (M) . Thus the operator X ^ DX satisfies equations , it follows that DxY = VxY for all X , Y e Vect (M) . This completes the second proof of (ii) .

We prove (iii) . Since 0 preserves the first fundamental form it also preserves the energy of curves , namely

E (0 ∘ Y) = E (Y)

for every smooth map 7 : [0 , 1] ^ M . Hence 7 is a critical point of the energy functional if and only if 0 o 7 is a critical point of the energy functional . Alternatively it follows from (ii) that

for every smooth curve 7 : I ^ M . If 7 is a geodesic the last term vanishes and hence 0oy is a geodesic as well . As a third proof we can deduce (iii) from the formula 0 (expp (v)) = exp^p) (d0 (p) v) in the proof of Theorem We prove (iv) . For to e I and vo e TY (t0) M define

X (t) := T7 (t , to) vo , X' (t) := (t , to) # (7 (to)) vo .

By (ii) the vector fields X' and 0*X along 0 o 7 are both parallel and they agree at t = to . Hence X' (t) = 0*X (t) for all t e I

We prove (v) . Fix a smooth map 7 : r2 ^ M and a smooth vector field Z along y , and define 7' = 0 o 7 : r2 ^ M' and Z' := 0*Z G Vect (Y') . Then it follows from (ii) that

R' (ds y ' , 3tY) z' = vytz> — yvs z'

= 0* (VSVZ — VtVsZ)

= d0 (Y) R (dsY , dtY) Z = (0* R) (dsY' , dtY') Z' .

7.5 GAUSSIAN CURVATURE

As a special case we shall now consider a hypersurface M C rm + 1 , i . e . a smooth submanifold of codimension one . We assume that there is a smooth map v : M ^ rm + 1 such that , for every p G M , we have v (p) T TpM and |v (p) | = 1 . Such a map always exists locally . Note that v (p) is an element of the unit sphere in rm + 1 for every p G M and hence we can regard v as a map from M to Sm:

v : M - >■ Sm .

Such a map is called a Gaufi map for M . Note that if v : M ^ S2 is a GauB map so is —v , but this is the only ambiguity when M is connected . Differentiating v at p G M we obtain a linear map

dv (p) : TpM ^ Tv (p) Sm = TpM

Here we use the fact that Tv (p) Sm = v (p) x and , by definition of the GauB map v , the tangent space of M at p is also equal to v (p) x . Thus dv (p) is a linear map from the tangent space of M at p to itself .

Definition . The Gaussian curvature of the hypersurface M is the real valued function K : M ^ r defined by

K (p) := det (dv (p) : TpM ^ TpM)

for p G M . (Replacing v by —v has the effect of replacing K by (—1) mK; so K is independent of the choice of the Gaufi map when m is even .)

Remark . Given a subset B C M the set v (B) C Sm is often called the spherical image of B . If v is a diffeomorphism on a neighborhood of B the change of variables formula for an integral gives

Ts = | K\tm

■JB

Jv (B) JB

where pM and ps denote the volume elements on M and Sm , respectively . Introducing the notation AreaM (B) := fB pM we obtain the formula

Notes

|K (p) | = lim A=ITf) ! .

B^p AreaM (B)

This says that the curvature at p is roughly the ratio of the (m - dimensional) area of the spherical image v (B) to the area of B where B is a very small open neighborhood of p in M . The sign of K (p) is positive when the linear map dv (p) : TpM ^ TpM preserves orientation and negative when it reverses orientation .

Remark . We see that the GauBian curvature is a natural generalization of Euler's curvature for a plane curve . Indeed if M C r2 is a 1 - manifold and p e M we can choose a curve 7 = (x , y) : (—e , e) ^ M such that y (0) = p and \j (s) \ = 1 for every s . This curve parametrizes M by the arclength and the unit normal vector pointing to the right with respect to the orientation of y is v (x , y) = (y , —X) . This is a local GauB map and its derivative (y , —X) is tangent to the curve . The inner product of the latter with the unit tangent vector j = (X , y) is the GauBian curvature . Thus

dx d2y dy d2x dd K := —— -

ds ds2 ds ds2 ds

where s is the arclength parameter and d is the angle made by the normal (or the tangent) with some constant line . With this convention K is positive at a left turn and negative at a right turn .

Exercise . The GauBian curvature of a sphere of radius r is constant and has the value r - m .

Exercise . Show that the GauBian curvature of the surface z = x2 — y2 is —4 at the origin .

We now restrict to the case of surfaces , i . e . of 2 - dimensional submanifolds of r3 . The difference between positive and negative GauBian curvature in dimension two .

K > 0   K = 0   K < 0

Theorem (GauBian curvature) . Let M C r3 be a surface and fix a point p <= M . If u , v <= TpM is a basis then

(R (u , v) v , u) |u|2 |v|2 — (u , v) 2'

Moreover , for all u , v , w <= TpM , we have

R (u , v) w = —K (p) (v (p) , u x v) v (p) x w .

Proof . The orthogonal projection of r3 onto the tangent space TpM = v (p) is given by the 3 x 3 - matrix

n (p) = 1 — v (p) v (p) T . dn (p) u = —v (p) (dv (p) u) T — (dv (p) u) v (p) T .

Here the first summand is the second fundamental form , which maps TpM to TpMx , and the second summand is its dual , which maps TpMx to TpM . Thus

hp (v) = v (p) (dv (p) v) T : TpM ^ TpM± , hp (u) * = (dv (p) u) v (p) T : TpM± ^ TpM .

By the GauB - Codazzi formula this implies

Rp (u , v) w = hp (u) *hp (v) w — hp (v) *hp (u) w

= (dv (p) u) (dv (p) v) Tw — (dv (p) v) (dv (p) u) Tw = (dv (p) v , w) dv (p) u — (dv (p) u , w) dv (p) v

and hence

(Rp (u , v) w , z) = (dv (p) u , z) (dv (p) v , w) — (dv (p) u , w) (dv (p) v , z) .

Now fix four tangent vectors u , v , w , z G TpM and consider the composition

r3 —4 r3 —4 r3 —4 r3

of the linear maps

:= ilv (p) + C2u + C3v ,

dv (p) n , if n A v (p) , n , if n g rv (p) ,

<Z , v (p))

CZ := I <C , z)

(C , w)

This composition is represented by the matrix

/ 1 0 0

CBA = I 0 <dv (p) u , z) <dv (p) v , z)

\ 0 <dv (p) u , w) <dv (p) v , w)

Hence ,

<Rp (u , v) w , z) = det (CBA)

= det (A) det (B) det (C)

= <v (p) , u x v) K (p) <v (p) , z x w)

= —K (p) <v (p) , u x v) <v (p) x w , z) .

This implies and

<Rp (u , v) v , u) = K (p) <v (p) , u x v) 2 = K (p) |u x v|2 = K (p) (|u|2 |v|2 — <u , v) 2) .

This proves Theorem□

Corollary (Theorem Egregium of Gaufi) . The Gaussian curvature is intrinsic , i . e . if f : M 4 M' is an isometry of surfaces in r3 then

K = K' o f : M 4 r .

Exercise . For m = 1 the GauBian curvature is clearly not intrinsic as any two curves are locally isometric (parameterized by arclength) . Show that the curvature K (p) is intrinsic for even m while its absolute value | K (p) | is intrinsic for odd m > 3 . Hint: We still have the equation which , for z = u and v = w , can be written in the form

, , , . ({dv (p) u , u) {dv (p) u , v)

(R„ (u , v) v . v) =de^ {dv (p) t , M) {dv (p) „ , v)

Thus , for an orthonormal basis v\ , . . . , vm of TpM , the 2 x 2 minors of the matrix

({dv (p) vi , vj)) i , j=1 , . . . , m are intrinsic . Hence everything reduces to the following assertion .

Lemma . The determinant of an m x m matrix is an expression in its 2 x 2 minors if m is even; the absolute value of the determinant is an expression in the 2 x 2 minors if m is odd and greater than or equal to 3 .

The lemma is proved by induction on m . For the absolute value , note the formula

det (A) m = det (det (A) 1) = det (AB) = det (A) det (B) for an m x m matrix A where B is the transposed matrix of cofactors .

Curvature in Local Coordinates*

Riemann

Let M c rk be an m - dimensional manifold and let

f = f) ~l : U ^ Q

be a local coordinate chart on an open set U c M with values in an open set Q c rm . Define the vector fields E1 , . . . , Em along by

Ei (x) := (x) e

These vector fields form a basis of T^;x) M for every x e Q and the coefficients gij : Q ^ r of the first fundamental form are

9ij = {Ei , Ej) . Recall from Lemma that the Christoffel rj : Q —> r are the coefficients of the Levi - Civita connection , defined by

m

vtEj = Y rkj Ek

k=l

and that they are given by the formula

m

r j := Y gk<= 2 (di9ie + dj 9i<= - di9ij) •

<==i

Define the coefficients

Rjk : Q ^ r

of the curvature tensor by

m

R{Ei , Ej) Ek = Y R<=jkE<= .

<==1

These coefficients are given by

m

Rj := di j - dj rfk + Y (rfv j j rk) .

V=1

The coefficients of the Riemann curvature tensor have the symmetries

Rijk<= Rjik<= Rij<=k Rk<=ij , Rijk<= : ^ ^ Rijkgv<= ,

V

and the first Bianchi identity has the form

R<=jk + Rj<=ki + Rkij = ° .

Warning: Care must be taken with the ordering of the indices . Some authors use the notation Rekij for what we call Rjk and R<=kij for what we cah Rijk<= .

If M c rn is a 2 - manifold (not necessarily embedded in r3) we can use equation as the definition of the GauBian curvature

K : M ^ r .

Let ^ : Q ^ U be a local parametrization of an open set U c M defined on an open set Q c r2 . Denote the coordinates in r2 by (x , y) and define the functions E , F , G : Q ^ r by

E := \dx^\2 , F := {dx^ , dy^) , G := \dy^|2 .

We abbreviate

D := EG - F2 .

Then the composition of the GauBian curvature K : M ^ r with the parametrization ^ is given by the explicit formula

1 / E F dyF — 1 dxG

K o ^ = — det I F G 1 dyG

\ 1 dXE dXF — 22 dyE — 1 dy2E + dxdyF — 1 dXG

1 I E F 2 dyE — det I F G 2dxG 1dyE 2 dxG 0

1 d (EdxG — FdyE

2\[Ddx V ^\ / D

1 d / 2EdxF — FdxE — EdyE

+ -

2VD dy\ eVD

This expression simplifies dramatically when F = 0 and we get TX i

1 V d dxG d dyE \

o ^ = —^E^ (dX TEG + oy TEG) (. .)

Exercise . Prove that the Riemannian metric

4

E = G = JUT 2w t\2 , F = 0

(1 + x2 + y2) 2

on r2 has constant constant curvature K = 1 and the Riemannian metric

Notes

4

$E = G = (1 - x^2 - y^2)^2 , F = 0 ,$

on the open unit disc has constant curvature $K = -1$.

Check your Progress - 1

Discuss Curvature

Discuss Theorem of Egregium

7.6 LET US SUM UP

In this unit we have discussed the definition and example of Curvature , Generalized Theorem of Egregium , Theorem of Egregium , Gaussian Curvature

7.7 KEYWORDS

Curvature ….. This chapter begins by introducing the notion of an isometry

Generalized Theorem of Egregium ….. We will now show that Geodesics , covariant differentiation , parallel trans - port , and the Riemann curvature tensor are all intrinsic

Theorem of Egregium ….. The first fundamental form , covariant differentiation , Geodesics , parallel transport , and the Riemann curvature tensor are intrinsic . This means that for every isometry 0 : M ^ M' the following holds .

Gaussian Curvature….. As a special case we shall now consider a hypersurface M C rm + 1 , i . e . a smooth submanifold of codimension one

7.8 QUESTIONS FOR REVIEW

Explain Curvature ,

Explain Theorem of Egregium

7.9 ANSWERS TO CHECK YOUR PROGRESS

Curvature (answer for Check your Progress - 1 Q)

Theorem of Egregium (answer for Check your Progress - 1 Q)

7.10 REFERENCE

Differential Geometry, Differential Geometry & Application, Introduction to Defferential Geometry, Basic of Differential Geometry.